

# Rethinking geometrical exactness

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This paper was written in homage to Henk Bos in recognition of all that he has taught me and pressed me to think about.  
It is a delayed part of the special issue of *Historia Mathematica* (37/3) dedicated to him.

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## Abstract

A crucial concern of early modern geometry was fixing appropriate norms for deciding whether some objects, procedures, or arguments should or should not be allowed into it. According to Bos, this is the exactness concern. I argue that Descartes's way of responding to this concern was to suggest an appropriate conservative extension of Euclid's plane geometry (EPG). In Section 2, I outline the exactness concern as, I think, it appeared to Descartes. In Section 3, I account for Descartes's views on exactness and for his attitude towards the most common sorts of constructions in classical geometry. I also explain in which sense his geometry can be conceived as a conservative extension of EPG. I conclude by briefly discussing some structural similarities and differences between Descartes's geometry and EPG.

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## Résumé

Une question cruciale pour la géométrie à l'âge classique fut celle de décider si certains objets, procédures ou arguments devaient ou non être admis au sein de ses limites. Selon Bos, c'est la question de l'exactitude. J'avance que Descartes répondit à cette question en suggérant une extension conservatrice de la géométrie plane d'Euclide (EPG). Dans la Section 2, je reconstruis la question de l'exactitude ainsi que, selon moi, elle se présentait d'abord aux yeux de Descartes. Dans la Section 3, je rends compte des vues de Descartes sur la question de l'exactitude et de son attitude face aux types de constructions plus communes dans la géométrie classique. Je montre aussi en quel sens sa géométrie peut se concevoir comme une extension conservatrice de EPG. Je conclus en discutant brièvement certaines analogies et différences structurales entre la géométrie de Descartes et EPG.

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2000 MSC: 01-02; 01A20; 01A45

**Keywords:** Euclid's plane geometry; Early modern geometry; Descartes's geometry

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## 1. Introduction

A crucial concern of early modern geometry was that of fixing appropriate norms for deciding whether some objects, procedures, or arguments should or should not be allowed in it. Henk Bos has devoted his main book to this concern, which he understands as an “endeavour to clarify and institute exactness” [Bos 2001, 3]. It focuses especially on Descartes and the way he changed the “concept of construction.” It is, to my mind, the most valuable study ever written on Descartes’s geometry and its historical framework. Though I widely agree with Bos’s insight, I would like to discuss and partially challenge some of his views.

Descartes’s geometry is a conservative extension of Euclid’s, and providing this extension is Descartes’s way of responding to the exactness concern. This is the main thesis I shall defend. To introduce it, let me offer a preliminary clarification in three points.

The first point concerns what I mean by ‘Euclid’s geometry’. This is the theory expounded in the first six books of the *Elements* and in the *Data*. To be more precise, I call it ‘Euclid’s plane geometry’, or ‘EPG,’ for short.<sup>1</sup> It is not a formal theory in the modern sense, and, *a fortiori*, it is not, then, a deductive closure of a set of axioms. Hence, it is not a closed system, in the modern logical sense of this term. Still, it is not a simple collection of results, nor a mere general insight. It is rather a well-framed system, endowed with a codified language, some basic assumptions, and relatively precise deductive rules. And this system is also closed in another sense [Jullien 2006, 311–312], since it has sharp limits fixed by its language, its basic assumptions, and its deductive rules. In what follows, especially in Section 2, I shall better account for some of these limits, namely for those relative to its ontology. More specifically, I shall describe this ontology as being composed of objects available within this system, rather than objects that are required or purported to exist by force of the assumptions that this system is based on and of the results proved within it. This makes EPG radically different from modern mathematical theories (both formal and informal). One of my claims is that Descartes’s geometry partially reflects this feature of EPG.<sup>2</sup>

Both in the early modern age and earlier, EPG was the subject of many critical discussions. Still, these were generally not aimed at questioning it, but rather focused on its interpretation, assessment, and systematization. One might even say that EPG constitutes the unquestioned core of classical geometry (as I suggest calling pre-Cartesian geometry as a whole). A crucial concern of classical geometry was extending EPG, that is, looking for appropriate ways to do geometry outside its limits. These efforts did not produce a closed

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<sup>1</sup> My restriction to plane geometry is not intended to imply that plane geometry is sharply distinct from solid geometry for Euclid, or that this is so for the successive mathematicians up to Descartes. This restriction merely depends on the fact that some of the claims I shall make about plane geometry would apply to solid geometry only in the case of a number of appropriate specifications, disclaimers, or adjustments. As the consideration of plane geometry is enough for my purpose, I prefer then to limit myself to it, for simplicity. For a number of relevant considerations of the matter of the relation between plane and solid geometry, I refer the reader to an ongoing paper of A. Arana and P. Mancosu (whom I thank for sending me some preliminary versions of their paper).

<sup>2</sup> If I say that Descartes’s geometry reflects this feature of EPG only partially, it is because of reasons that depend on Descartes’s geometrical algebra. This is a quite crucial matter, of course, but it is not directly relevant for my present purpose. Hence, I shall not consider it in the body of my paper, and only get to it briefly in the Concluding Remarks (Section 4).

(in the sense specified above) and equally well-framed system as EPG, however, so that classical geometry appears neither as a single theory, nor as a family of theories, but rather as quite a fluid branch of studies. This is my second point.

My third and main point is that things are quite different with Descartes's geometry: this is a closed system, equally well-framed as EPG. Descartes not only took EPG for granted [Jullien 1996, 10–11] and based his own geometry on it, but also grounded this last geometry on a conception of the relations between geometrical objects and constructions that is structurally similar to that which pertains to EPG. This is the sense in which I say that Descartes's extension of EPG is conservative.<sup>3</sup>

Despite this strict connection between Descartes's geometry and EPG, many accounts of the former emphasize its novelties and differences with respect to classical geometry and project it towards its future, rather than rooting it in its past. Though this is not the case for Bos's book<sup>4</sup> (which also emphasizes the conservative nature of Descartes's geometry [Bos 2001, 411–412]), it seems to me that something more should be said on the relation between Descartes's geometry and classical geometry.

I hold that Descartes's geometry is better understood if its structural affinities with EPG are pointed out. Also, if this is done, its crucial novelties may, at least partially, be accounted for as the quite natural outcome of an effort to extend EPG and get a closed and equally well-framed system.

When things are viewed this way, Descartes's primary purpose in geometry appears to be a foundational one, and his addressing the exactness concern appears as a crucial ingredient of this purpose. This does not appear to be Bos's understanding. He rather maintains that the “primary aim” of the *Géométrie* “was to provide a general method for geometrical problem solving” [Bos 2001, 228]. I do not deny that solving geometrical problems was a pivotal concern for Descartes. Still, I argue that it naturally arose within his foundational program. I have just said (and I shall try to justify later) that Descartes's geometry partially reflects the crucial feature of EPG that I account for by describing its ontology as composed of objects available within it (rather than of objects that are required or purported to exist). In my mind, this is enough to explain why Descartes was mainly concerned with the solution of geometrical problems without arguing that the *Géométrie* was primarily written to present a method for geometric problem-solving (however suitable and general this method could have appeared to him).

This view has another important consequence. The “general method for geometrical problem solving” that Bos refers to is certainly not to be confused with the method “of correctly conducting one's reason and seeking truth in the sciences” that the *Géométrie* is famously supposed to be an essay of Descartes [1637]; Descartes [AT], IV; Descartes [DMML]. This last method is, even more famously, based on the clarity and distinctness precept [Descartes 1637, 20, 34, 39; Descartes [AT], VI, 18, 33, 38; Descartes [DMML], 17, 29, 33]. This makes it quite natural to think that Descartes's concern for geometrical exactness is the geometrical aspect of his quest for clarity

<sup>3</sup> This sense is highly informal. In Section 2.3, I shall clarify it a little bit more. In footnote 68, I shall briefly consider, instead, the question of whether Descartes's geometry can be said to be a conservative extension of EPG in a closer sense to the technical one which is usual in modern logic.

<sup>4</sup> Another notable exception is a classical paper of Molland [1976]. Despite many local affinities and a common emphasis on foundations, the views defended in this paper are, however, quite different from those I shall argue for.

and distinctness.<sup>5</sup> Bos does not explicitly endorse this thesis, but makes a very similar claim that would imply it if it were admitted (as it is natural to do) that, for Descartes, clarity and distinctness are necessary ingredients of the quest for truth and certainty. Namely, Bos argues that “for Descartes the aim of methodical reasoning was to find truth and certainty,” and that “in geometrical context this quest concerned what I refer to by the term ‘exactness’” [Bos 2001, 229]. I cannot discuss this matter here. But I nevertheless observe that if, for Descartes, exactness were the geometrical counterpart of clarity and distinctness, then clarity and distinctness in geometry could not be merely a matter of rational conceivability, but should be strictly intertwined with the satisfaction of constructive requirements that, as I shall show later, directly derive from Euclid’s.

The present Introduction aside, the paper includes two main sections followed by some Concluding Remarks.

Section 2 outlines the exactness concern with respect to classical geometry, and accounts for the way it would have appeared to Descartes. For this purpose, I come back to some known material that has been enlighteningly analyzed by Bos, by emphasizing some aspects of it. In Section 2.1, I distinguish exactness from precision, and also introduce some terminology that I shall use later. Section 2.2 is devoted to EPG, by particularly emphasizing the role that problems and constructions have in it. This role explains why a conservative extension of EPG requires the admission of new sorts of constructions and new tools for solving problems. Section 2.3 offers different examples of the way classical geometry extended EPG. This allows me to distinguish six different sorts of constructions not admitted within EPG.

Section 3 accounts for Descartes’s views on exactness.<sup>6</sup> The matter is introduced in Section 3.1, by considering Descartes’s attitude to the mean proportionals problem. Section 3.2 then provides a systematic account of his characterization of geometrical curves, whereas Section 3.3 accounts for Descartes’s different attitudes toward the six different sorts of constructions distinguished in Section 2.3.

Finally, the concluding Section 4 briefly accounts for some structural similarity and essential differences between Descartes’s geometry and EPG.

## 2. The exactness concern

### 2.1. Exactness norms

The exactness concern for classical geometry was not a matter of accuracy. Accuracy was certainly a requirement for practical or applied geometry, but the exactness requirement

<sup>5</sup> The conceptual and methodological relations between the *Discours de la Méthode* and the *Géométrie* are far from simple and I cannot enter into this matter here. I only observe that in Descartes’s correspondence one finds evidence for arguing both that he considered them to be strictly connected, and that he took them to be relatively independent. As examples, I quote a passage from a letter to Mersenne from the end of December 1637 and another from a letter to Vatier from February 22th 1638. “In the *Dioptrique* and the *Météores* I merely tried to persuade [someone] that my Method is better than the usual one; in my *Géométrie*, however, I claim to have demonstrated this” [Descartes [AT], I, 478; I slightly modify the translation of *The Philosophical Writings of Descartes*: Descartes [PWC], 77–78]. “I could not demonstrate the use of [...] my method in the three treatises that I gave, because it prescribes an order for searching things which is quite different from that I thought to have to use for explaining them” [Descartes [AT], I, 559; again, I slightly modify the translation of *The Philosophical Writings of Descartes*: Descartes [PWC], 85].

<sup>6</sup> On this matter, I also refer the reader to Panza [2005, 23–43].

concerned pure geometry, and was quite different<sup>7</sup>: whereas, for the purposes of practical geometry, it was required to perform some (material) procedures with a sufficient degree of precision, in pure geometry it was required to argue in some licensed ways. This is what the exactness concern was about. In order to better explain this matter, I need a convenient terminology.

I use the term ‘concept’ for short, to refer to what should be more precisely called ‘sortal concepts’. In philosophical literature, there is no general agreement about the intrinsic nature of concepts. Still, it is widely admitted that, whatever concepts might be, sortal concepts should be such that the assertion that some objects fall or do not fall under them is meaningful. Moreover, it is also widely admitted that each sortal concept is characterized if and only if two kinds of conditions are attached to it: its application conditions and the identity conditions of the objects that are purported to fall under it. The former are necessary and sufficient conditions for an object to fall under this concept: an object meets them if and only if it falls under this concept. The latter are necessary and sufficient conditions for the objects that fall under this concept to be distinct from each other: if *a* and *b* are objects that fall under it, *a* is the same object as *b* if and only if these conditions are met.<sup>8</sup>

Fixing the former kind of conditions for a certain concept is possibly not enough to fix the latter kind of conditions. Moreover, this is possibly not enough to provide either a warrant that some objects actually fall under this concept or appropriate norms for obtaining some objects which do. Again, fixing both kinds of conditions is possibly not sufficient to

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<sup>7</sup> I emphasize that the question concerns pure geometry in order to make the distinction between accuracy and exactness clearer. From now on, I shall avoid this specification and take it for granted that ‘geometry’ refers to pure geometry.

<sup>8</sup> A simple example can be useful to better explain this notion. Suppose that somebody, let us say Ann, is looking to the sky and is willing to describe what she is seeing. She would presumably appeal to the sortal concept of star. She would then say that some of the things she is seeing are stars, i.e., objects that fall under this concept. If Ann wanted to be really precise in what she is saying, she should be able to explain what makes something a star. Stating it would be the same as stating the application conditions of the concept of star. But this would still not be enough, since Ann should also be able to explain what makes one of the stars that she is seeing now the same as one of those that she was seeing yesterday. Stating it would be the same as stating the identity conditions of stars. If Ann were actually able to do both things, she would possess the sortal concept of star. But suppose now that, among the things that Ann is seeing, there was one that she cannot recognize very well. She could then wonder whether this was a flying saucer. Regardless of whether she would conclude that this was so or not, in order to be really precise in what she was thinking, Ann should also have in mind appropriate application conditions for the concept of flying saucer, and possibly also identity conditions for flying saucers. The appropriateness of these conditions would not depend of course on the actual existence of flying saucers (more than that, if these conditions, or at least the former of them, were not appropriate, it would be impossible to rightly conclude that flying saucers do not exist, in fact). This should be enough for making clear that it is not necessary that some objects actually fall under a sortal concept for a certain sortal concept to be clearly identified as such. What matters is only that the assertion that some objects fall or do not fall under it be meaningful and that the application and identity conditions be fixed, as said above. A last remark, for completeness. The question of whether and how one could appropriately distinguish sortal concepts from nonsortal ones is quite complex, and philosophical literature displays no general agreement about it. But this does not matter for my purpose, since I use the term ‘concept’ only to refer to sortal concepts, as I have clarified above.

provide this warrant and these norms. This sort of independence is realized in the case of the geometrical concepts involved in classical geometry.

Consider an example. For the sake of simplicity, it pertains to EPG, but it is evoked here to account for some features of classical geometry as a whole. What I shall say about it in the present Section is thus intended to apply, *mutatis mutandis*, to all classical geometry. I shall specifically consider EPG in the following Section 2.2.

Definitions I.19–20 of the *Elements* fix the application conditions of the concept of equilateral triangle. They do it by stating that equilateral triangles are rectilinear figures contained by three equal segments<sup>9</sup> (I understand this statement this way: an object falls under the concept of equilateral triangle if and only if it is a rectilinear figure contained by three equal segments). These same definitions do not provide, however, identity conditions for the objects that possibly fall under this concept. Moreover, they provide neither a warrant that some objects actually fall under it nor appropriate norms for obtaining some such objects. One might then doubt that these definitions would be enough to define equilateral triangles.

As a matter of fact however, in classical geometry, fixing the application conditions of a certain concept—a geometrical concept of course (that is, a concept under which geometrical objects are purported to fall)—was considered to be enough to define such objects. For example, Definitions I.19–20 of the *Elements* were taken to be enough to define equilateral triangles (in what follows, I will conform with this attitude and use accordingly the verb ‘to define’ and its cognates).

Once a definition such as this was offered, it was still necessary to provide the identity conditions of the relevant objects, and a warrant that some objects fall under the relevant concept, and/or some norms for obtaining some such objects. Returning to our example, one might think that providing this warrant in this case would have been the same as ensuring that the equilateral triangles exist. But this is not so in fact, at least if it is admitted (as it is usual in modern mathematics) that, for a concept *P*, the *Ps* exist only insofar as they form a fixed domain of quantification and individual reference. This means that admitting (or supposing) that the *Ps* exist is taken to license both asserting that some appropriate conditions are obtained for all of them taken individually (for example, that all, taken individually, enjoy a certain property), and denoting each one, or at least some of them, with an appropriate singular term that rigidly refers to it (which entails that appropriate identity conditions are available for them).<sup>10</sup>

This way of conceiving the existence of objects of a certain sort is quite natural, I think. It is then natural to assert that the equilateral triangles that EPG is about (i.e., the objects that are purported to fall under the concept whose application conditions are fixed by Definitions I.19–20 of the *Elements*) do not exist.<sup>11</sup> A simple reflection should convince us of that.

<sup>9</sup> For my use of the term ‘segment,’ cf. footnote 15.

<sup>10</sup> A simple example of this way of thinking is the following: admitting (or supposing) that the natural numbers exist is taken to license both asserting that they form a progression, and denoting one of them with the term ‘1’ which refers to the same natural number in any one of its occurrences (note that saying that asserting that *p* is licensed does not mean that *p* is warranted to be true, but just that it is ensured that the assertion that *p* is meaningful).

<sup>11</sup> Notice that to say that the *Ps* do not exist is not the same as saying that no particular object falling under *P* can exist, or that no such object exists in a certain context. By ‘the *Ps* do not exist’ I merely mean that there is nothing like a definite totality of all the *Ps* in the sense just explained. This point will become clearer, I hope, on the basis of the following considerations.



Imagine two historians of mathematics who, on two distinct occasions, relate the solution of Proposition I.1 of the *Elements*, which asks one to construct an equilateral triangle on a given segment. Suppose now that somebody were asking whether they, by doing this, were speaking of the same equilateral triangle. More generally, suppose that such a somebody were asking under which conditions one could speak of the same equilateral triangle on different occasions, by relating some EPG arguments. It seems obvious to me that both these questions are ill-posed, since EPG runs perfectly even if no way to answer them is provided. More than that, there is no clear sense in which one could fix the reference of a singular term for equilateral triangles in the language of EPG (for example, ‘ABC’) in such a way that it be taken to refer to the same equilateral triangle in any one of its occurrences. For this same reason, it is inappropriate to assert that all the equilateral triangles that EPG is about, taken individually, enjoy a certain property.<sup>12</sup>

What does it mean, then, in this as in other cases relative to classical geometry, that a warrant that some objects actually fall under a certain concept (appropriately defined) is provided? This means that it has been shown how to put one or several such objects (distinguished from each other) at the disposal of a mathematician doing geometry for the purpose of producing an argument about them. Now, in classical geometry, this is done in such a way that it makes no sense to wonder whether the objects of this sort that a certain argument is about are or are not the same as other objects that another, independent argument is about.

Consider our example again. The warrant that some objects actually fall under the concept of equilateral triangle is provided by the solution of Proposition I.1 of the *Elements*. I shall better explain this point later, since this explanation requires a distinction between two kinds of geometrical concepts involved in EPG that I have not introduced yet. For the time being, the only relevant point is that this solution exhibits a procedure, namely a construction, that applies to any given segment and results in an equilateral triangle having this segment as a side. Hence, if a segment is at the disposal of a mathematician doing EPG, it is enough for her/him to apply this procedure in order to also have at her/his disposal an equilateral triangle (having this segment as a side). In my parlance, this ensures that some objects fall under the concept of equilateral triangle, though this does not prove, of course, that equilateral triangles exist in the sense explained above. I also doubt that there is some other clear sense in which one can say that this proves that the equilateral triangles exist. At most, after having constructed it, one can say that a particular equilateral triangle is brought into existence. But then, one should also admit that this triangle exists only in the context of the argument in which this construction is involved, since no clear condition is provided for ensuring that this same triangle also occurs in another, independent argument.

The same happens for the objects falling under any other geometrical concept involved in classical geometry. For short, I use the verb ‘to obtain’ to mean the action of putting some objects that fall under a certain concept (whose application conditions have been appropriately fixed) at the disposal of a mathematician for the purpose of producing an argument about them. This explains what I mean by speaking, with respect to classical geometry, of norms for obtaining such objects. They are norms that the procedures to be followed for obtaining these objects (that is, for putting them at the disposal of a mathematician for the purpose of producing an argument about them) have to comply with in order

<sup>12</sup> This does not mean of course that EPG does not include universal statements about certain sorts of objects. The contrary is true: theorems in EPG are just such statements. Still, according to me, a theorem in EPG does not state that all the objects of a certain sort, taken individually, enjoy a certain property. I shall better explain this point at the end of Section 2.2.

to be licensed. Hence, in order to provide the warrant that some objects actually fall under a certain concept, one has to show how to obtain some objects that fall under this concept through a procedure that complies with these norms.

Consider, once more, our example. The procedure exhibited by the solution of Proposition I.1 is licensed within EPG because it obeys some constructive clauses explicitly stated in the *Elements*, and takes advantage, in a way that is implicitly allowed, of the physical properties of the relevant diagrams. It is then licensed because it complies with some norms explicitly stated or implicitly admitted in the *Elements*. These are norms for obtaining some objects falling under a certain concept. Namely, they are those that are proper to EPG.

In my view, the exactness concern for classical geometry was essentially that of providing norms such as these. This is why I suggest calling them ‘exactness norms’.

Those proper to EPG were quite clearly identified. But this was not so for classical geometry in general. Not only did different geometers adopt quite different such norms, but also very often these norms were left implicit, or made explicit in a way open to different understandings. One of the aspects under which Descartes’s geometry is an extension of EPG is that it includes appropriate exactness norms licensing obtaining some geometrical objects that cannot be obtained according to the exactness norms proper to EPG. The way that these norms are stated is also open different understandings, however. Bos’s book is a major contribution to the effort of fixing the most appropriate and plausible understanding. In the present paper, especially in Section 3, I shall also try to contribute to this effort.

## 2.2. Problems in Euclid’s plane geometry

In EPG, problems ask for constructions, and they are solved insofar as these constructions are performed. These are constructions of objects that are required to fall under some specified concepts. More precisely, in EPG each problem is a demand that one or more objects purporting to fall under one or more concepts be constructed. For short, I say that a problem which asks for the construction of an object falling under a certain concept is concerned with this concept. The main purpose in stating problems in EPG is thus, in my parlance, that of providing an appropriate specification of the concepts they are concerned with.

The objects that EPG is about, or EPG objects, as I shall say from now on, are points, segments, circles, angles,<sup>13</sup> and polygons of distinct sorts.<sup>14</sup> The relative concepts are intro-

<sup>13</sup> I use the term ‘angle’ to refer, in general, to rectilinear angles or angles formed by a segment and a circle or two circles. For short, when I use this term to refer to a particular angle, I shall intend that it is a rectilinear one.

<sup>14</sup> The objects that EPG is about, or EPG objects, are, of course, objects that are purported to fall under concepts whose application conditions may be fixed using the language of EPG. The inverse implication does not hold, however. A simple example is enough to explain why. According to Definition I.15 of the *Elements*, “a circle is a plane figure contained by a line such that all the straight lines falling upon it from one point among those lying inside such a figure are equal to one another” (I slightly modify Heath’s translation [Euclid [ECH], I, 153 and 183]). It is enough to slightly modify this definition in order to define ellipses (or to fix the application conditions of the concept of ellipse). It is clear however that ellipses are not objects that EPG is about. Something similar can also be said for parabolas and hyperbolas, and many other curves other than circles. I prefer then to say explicitly that EPG objects are points, segments, circles, angles, and polygons of distinct sorts. More precisely, according to my use of the term ‘EPG object,’ for a geometrical object to be an EPG one, it is enough that it is a point, a segment, a circle, an angle, or a polygon of some sort.



duced through explicit definitions that fix their application conditions. I have already offered an example above. This concerns the concept of equilateral triangle. To fix its application conditions, two other concepts are invoked: that of segment and that of figure (since a triangle is a polygon, and polygons are taken to be figures). These are introduced through Definitions I.2, I.4, and I.13–14 of the same *Elements*. These definitions are much less clear than Definitions I.19–20, and have been the object matter of innumerable comments and discussions. Still, for my present purpose, the subtleties involved in them do not matter. What matters is rather that, in order to state the application conditions of the concept of equilateral triangle, Euclid appeals to the concepts of segment and figure, and supposes it to be clear what is meant for three segments to be equal and to contain a figure.<sup>15</sup> This is a simple example of the practice of introducing a concept by appealing to other concepts already introduced. The concepts of point, segment, circle, angle, and other kinds of polygons are also introduced in this way.<sup>16</sup>

Some problems in EPG are concerned with concepts such as these. But this is not the case of all EPG problems.<sup>17</sup> Many of them (the majority of them, actually) are concerned, instead, with concepts that differ from these insofar as the objects that are purporting to fall under them are required to stay in some appropriate relation with other given objects. To appreciate the difference, compare, for example, the concept of square with that of square equal to a given rectangle or to a given circle. Of course, a square equal to a given rectangle or to a given circle is a square, but it should be immediately clear that asking for the construction of a square equal to a given rectangle or to a given circle is quite different from asking for the construction of any square (i.e., of a square having any arbitrary segment as its side).

<sup>15</sup> Definitions I.2, I.4, and I.13–14 are well known, but I quote them for the reader's benefit. According to Definition I.2, "a line is breadthless length"; according to Definition I.4, "a straight line is that which lies evenly with respect to the points on itself"; according to Definition I.13, "a boundary is that which is an extremity of anything"; finally, according to Definition I.14, "a figure is that which is contained by any boundary or boundaries" (I quote Heath's translations, slightly modifying that of Definition I.4 [Euclid [ECH], I, 153, 158, 165, and 182]). Only two simple remarks are appropriate for my present purpose. The former is that straight lines are generally finite, for Euclid, and this is the reason that I refer to them as segments. The latter is that the notion of being contained by something, which is involved in the definition of equilateral triangle, is already involved in the definition of figures, to the effect that, if Definitions I.13–14 are taken to be clear, what matters, in order to have a clear understanding of the definition of equilateral triangles, is admitting that three segments can provide a boundary or extremity.

<sup>16</sup> One might doubt that this is the case for the concept of point. To see that this is so, it is enough to remark that Definition I.1 is not enough to fix the application conditions of this concept. Definition I.3 is also required for that. The former states that "a point is that of which there is no part"; the latter clarifies this statement, by stating that "the extremities of a line are points" (I quote again Heath's translations, slightly modifying that of Definition I.1 [Euclid [ECH], I, 153, 155, and 165]).

<sup>17</sup> From now on, I call 'EPG problems' those problems that ask for constructions of EPG objects, by requiring (often implicitly) that these construction comply with the exactness norms proper to EPG. A usual and compact way to identify these norms consists in saying that they select constructions by ruler and compass. In this parlance, one could say that EPG problems are those problems that ask for constructions of EPG objects by ruler and compass. In what follows, I shall try to account for the exactness norms proper to EPG in a more precise way, and I shall also explain why I prefer to use a different terminology.

I shall come back to this distinction later. For the time being, it is only important to observe that—with only the exception of arbitrary segments and points, which provide the starting point of any construction licensed within EPG (as I shall explain later)—solving a problem is the only way available in EPG for warranting that some objects fall under a certain concept, independently of whether this concept is of the former kind or the latter. Construction is thus the typical modality through which geometrical objects are obtained in EPG, and to claim that some objects fall under a certain concept is, in EPG, the same as claiming that some objects falling under this concept can be constructed in the appropriate way.

Accordingly, the sense in which it can be said, in EPG, that some objects fall under a certain concept is manifested by the way in which EPG problems are solved. This also displays the identity conditions of EPG objects, and the exactness norms relative to them. A good way (the only way, in fact) to understand this sense and become aware of these conditions and norms is, then, by parsing the solutions of EPG problems. This is what I shall briefly do in the rest of the present section.

EPG constructions require that appropriate diagrams be drawn. More than that: they are just procedures for drawing diagrams in a licensed way, to the effect that an EPG problem is solved when appropriate diagrams, representing some objects falling under the concepts this problem is concerned with, are so drawn, or imagined to have been drawn. I term constructions like these ‘diagrammatic.’<sup>18</sup>

<sup>18</sup> Proposition I.1 of the *Elements* invoked above provides a very simple example. As written, it asks for the construction of an equilateral triangle on a given segment. In its solution, this segment is identified with that which is represented by a certain stroke. Nothing compels someone who is performing or expounding this solution to actually draw this stroke, of course. But the mere phrase ‘let AB be the given segment’ with which this solution begins (I quote Heath’s translation, slightly modifying it [Euclid [ECH], I, 241]) is understandable only insofar as a stroke representing a certain segment is imagined, if not actually drawn. This is because the way for identifying a single particular segment within EPG is by taking it to be the segment represented by a certain stroke. Hence, the stroke (either actually drawn or imagined) is needed to fix the reference of the singular terms ‘AB,’ ‘A,’ and ‘B’ [Netz 1999, 19–26]. Once this reference is fixed, the construction can begin. One “describes” two circles with radius AB: one with center A, the other with center B. Usually this goes with the actual drawing of two contour-closed lines passing respectively through the two extremities of the stroke representing the given segment, and representing these circles, in turn. Once more, this is not compulsory. But it is at least necessary to imagine these lines. This is all the more evident because the construction continues by observing that these circles meet in a point C, which, so to say, pops up, because of the physical properties of the lines that have been drawn or imagined. This point is, indeed, represented by the intersection of these lines, which provides the reference for the singular term ‘C.’ Then, this point is “joined” with A and B, respectively, which usually goes with the actual drawing of two new strokes that provide the actual reference of the new singular terms ‘CA’ and ‘CB.’ Again, these can be only imagined. But whether they are actually drawn or only imagined, they are needed in order to fix the reference of the other singular term that is supposed to denote the equilateral triangle that is constructed this way. What I refer to by the term ‘diagram’ is just the system composed of the three strokes representing the sides of this triangle and the two contour-closed lines whose intersection represents the point C. Of course, one can deny that Euclid’s solution involves three particular single segments, a particular circle, and a particular equilateral triangle, and then that ‘AB,’ ‘A,’ ‘B,’ etc. have to be understood as genuine singular terms. One can think, rather, that this solution concerns the very concepts of segment, circle, and equilateral triangle, or something like the corresponding schemas. Alternatively, one can think that ‘AB,’ ‘A,’ ‘B,’ etc. are genuine singular terms referring to abstract objects implicitly defined by the deductive rules that

The verb ‘to represent’ evokes a complex relation. I only emphasize here that diagrams provide the identity conditions of the objects they represent. This means that, within an argument concerned with several EPG objects, these are distinct insofar as they are represented, or supposed to be represented, by distinct diagrams or subdiagrams.<sup>19</sup>

A natural question arises then: how do diagrams differ from each other in EPG? It is not easy to answer, in general. For example, there is no clear response to the question of whether one can draw the same diagram twice, that is, whether diagrams are tokens or types.<sup>20</sup> I prefer to consider them as tokens. But, for my present argument, this is not essential. What matters is that considering them as tokens and admitting that, in a single diagram, different appropriate subdiagrams represent different objects allows for the conducting of EPG arguments (some possible unimportant exceptions apart).<sup>21</sup> In other terms, EPG works perfectly (these exceptions aside) if the diagrams involved in it are considered as tokens.<sup>22</sup> The only identity conditions that diagrams confer to EPG objects are thus local—that is, relative to single arguments—and no other identity conditions for geometrical objects are available in EPG. It follows that EPG objects do not form a fixed domain of quantification and individual reference, in the sense explained in Section 2.1: they are not

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terms like those submit to, and that diagrams enter into Euclid’s arguments only as a convenient but nonessential visual support of a completely independent syntax. I cannot argue here against these interpretations, which I think to be simply unfaithful to Euclid’s text and to the way it has been understood in classical geometry. I limit myself to adhering to another view (which is quite common, in fact), and to emphasizing the role that, according to this view, one has to confer to diagrams within EPG. For more details on this matter, I cannot but refer the reader to another paper of mine: Panza [TRD]. For a survey of the recent discussion on the role of diagrams in Euclid’s geometry updated to 2008, cf. also Manders [2008].

<sup>19</sup> For connoisseurs, I add that I take EPG objects to be quasi-concrete ones in Parsons’s sense [Parsons 2008, Sect. 7 and Ch. 5]. These are abstract objects “distinguished by the fact that they have an intrinsic relation to the concrete” (which he also calls ‘representation’), to the effect that they are “determined” by some concrete objects which provide “concrete embodiments” of them (ibid. 33). This does not entail that the identity conditions of quasi-concrete objects are those of their concrete counterparts. This is typical of EPG objects, in my view.

<sup>20</sup> In philosophical literature, the distinction types/tokens is, broadly speaking, that between an abstract object conceived as a general template, and its particular instances, the things that are taken to satisfy this template. A very nice example is offered by L. Wetzel in her article on this matter in the *Stanford Encyclopedia of Philosophy* (<http://plato.stanford.edu/entries/types-tokens/>). Take G. Stein’s verse in her poem *Sacred Emily*: ‘Rose is a rose is a rose is a rose.’ How many words are there in it? One can answer that there are three words: ‘rose,’ ‘is,’ and ‘a’. But one can also say that there are ten words, since ‘rose’ has four occurrences, and ‘is’ and ‘a’ three occurrences each. In the former case, one is counting types; in the latter, one is counting tokens. In the same way, one can also say that the concrete inscription of Stein’s verse that the reader has in front of her/his eyes is a token whose type is this verse itself. The question concerning EPG diagrams is then whether one should take the term ‘diagrams’ to refer to concrete inscriptions made on concrete sheets of paper or on some other supports, or to refer to some template of which these inscriptions are instances.

<sup>21</sup> Exceptions occur in some very particular cases such as when, for practical convenience, throughout the course of a single argument, a diagram is reproduced several times under the convention that it remains the same, or represents the same objects.

<sup>22</sup> Of course, nothing would forbid one to consider, for example, that each reformulation of the solution of Proposition I.1 involved different instances of the same diagram. The point is that this would be a useless convention, a convention that is not required for EPG to work. Hence, it would not be part of it, but imposed on it from outside.

pieces of basic furniture, somehow steady so that geometers might refer to them individually, using appropriate terms endowed with a rigid reference. They are merely objects that fall under some concepts and enter into particular arguments insofar they are represented, or supposed to be represented, by appropriate diagrams. And each time one wants to refer individually to some of them within a new argument, they have to be obtained, or to be supposed to have been obtained anew, by drawing, or imagining to draw, appropriate diagrams.

This is a crucial fact about EPG and makes it structurally different from modern mathematical theories, as anticipated in the Introduction. But what is even more relevant for my present purpose is that this makes it so that exactness norms are an essential ingredient of EPG. Insofar as EPG objects are obtained by construction, these are norms for performing constructions. Hence, to account for them, we have to consider how constructions work in EPG.

I have just said that constructions in EPG are diagrammatic. This means that the clauses they obey (i.e., the stipulations licensing their successive steps) are nothing but rules for drawing diagrams and for ascribing to them the power of representing certain geometric objects having certain relations to one another. Different systems of such clauses correspond to different sorts of diagrammatic constructions. The constructions entering into EPG obey a system of clauses such as these, either explicitly stated in the *Elements* (essentially through Postulates I.1, I.2, and I.3), or implicitly but systematically admitted.<sup>23</sup> These constructions are usually termed ‘by ruler and compass.’ Still, rulers and compasses do not occur in Euclid’s exposition. This name rather depends on a particular understanding which is an essential ingredient of Descartes’s geometry. Hence, I prefer to use it only for accounting for this last understanding. For a more neutral use, I suggest terming these constructions ‘elementary.’ Using this parlance, one can then say that EPG exactness norms reduce to a general requirement according to which an EPG object is obtained if and only if it is represented by a diagram that has been drawn, or at least imagined as having been drawn,<sup>24</sup> according to the clauses of elementary constructions.

This is equivalent to adding a supplementary condition to the problems advanced in EPG, a condition that characterizes EPG problems in general.<sup>25</sup> Hence, solving such a problem is more than constructing some appropriate objects; it is constructing these objects through an elementary construction. This is then more than providing a warrant that some

<sup>23</sup> The simple example considered in footnote 18 provides a clear illustration of some of these norms. Insofar as Proposition I.1 is the first proposition of the *Elements*, the mere fact that the construction entering into its solution starts with the admission that a stroke representing a segment is drawn (or imagined to be drawn) shows that constructions in EPG can begin with such an admission. This is a first implicit constructive clause (better, it is a particular case of a more general constructive clause to which I shall come back later). The second step in the construction—the description of the two circles of radius AB represented by two contour-closed lines passing respectively through the two extremities of this stroke—obeys another clause, explicitly stated by Postulate I.3. These contour-closed lines cut each other. A third constructive clause, implicit again, allows taking their intersection as representing a point that is *ipso facto* constructed insofar as these two circles are so. The last step—in which this point is joined respectively to the points A and B by two segments represented by two new strokes—obeys a further clause, explicitly stated by Postulate I.1. Finally, a last implicit clause allows taking the system of the three previous strokes as a representation of a triangles that is then proved to be equilateral.

<sup>24</sup> Cf. footnote 18.

<sup>25</sup> Cf. footnote 17.

objects fall under the relevant concepts; it is proving that objects falling under these concepts can be so constructed, that is, that they are available within EPG, as I suggest to say.<sup>26</sup>

To better clarify this point, one has to remark that, as anticipated above, EPG problems are concerned with two kinds of concepts. The former, which I term ‘unconditional,’ are such that the objects that fall under them are not required to stay in some appropriate relation with other given objects. This is the case, for example, of the concepts of point, segment, equilateral triangle, or square. The latter, which I term ‘conditional,’ are such that the objects that fall under them are required to stay in some appropriate relation with other given objects. This is the case, for example, of the concepts of point cutting a given segment in extreme and mean ratio, of segment perpendicular to another given segment, of equilateral triangle equal to another given triangle, of square equal to a given rectangle, or to a given circle. For short, I also term geometrical objects ‘unconditional’ or ‘conditional’ according whether they are purported to fall under unconditional or conditional concepts, respectively.

Suppose that  $P_U$  is an unconditional concept, for example, the concept of square, and  $P_C$  a conditional concept specifying  $P_U$  in some way, for example, the concept of square equal to a given circle. From the fact that unconditional objects falling under  $P_U$  are available within EPG it does not follow, of course, that conditional objects falling under  $P_C$  also are so. The opposite holds, instead: conditional objects falling under  $P_C$  are available within EPG only if unconditional objects falling under  $P_U$  are so. Hence, one could say that the basic ontology of EPG is formed by unconditional objects available within it, whereas the relational arrangement of this ontology depends on which conditional objects are available within it.

To appreciate the significance of this distinction it is essential to clarify what is meant in the language of EPG by saying that a certain object is given. The verb ‘to give [ $\delta\acute{\iota}\delta\omega\mu\iota$ ],’ especially its past participle ‘given’ (that is, the different forms of the aorist passive participle, in Greek), is typically used in the *Elements* to indicate the starting stage of a particular construction. If some objects are said to be given, a particular construction is licensed to start from them. This means that the diagrams representing the objects to be constructed have to be drawn starting from those that represent these objects. In the *Data*, ‘given’ is

<sup>26</sup> Harari has argued against Zeuthen’s classical “existential interpretation” of Euclid’s constructions [Harari 2003; Zeuthen 1896]. According to her, this interpretation assigns to Euclid three theses that he does not actually endorse, namely that: (i) “the correspondence between a defined term and the reality to which it refers cannot be taken for granted, but it rather should be established by means of proofs” [Harari 2003, 4]; (ii) geometrical constructions are “means of justification, i.e., [...] logical procedure[s] [...] aimed at establishing the truth-value of a given content” [ibid., 5]; (iii) they are also “means of ascertaining an already given content,” that is, “means of instantiating a universal concept [ibid., 14]. In opposition to (iii), Harari also argues that (iv) for Euclid, constructions are “positive means contributing content” [ibid.], both insofar as they are “means of measurement by which quantitative relations are deduced,” and as they exhibit or generate “spatial relations” [ibid., 1 and 21–22]. I agree that Euclid does not endorse (i)–(iii), if the terms “reality” and “content” in (i) and (iii), respectively, are taken to refer to something existing in the sense explained in Section 2.1, and this same last term in (ii) is taken to refer to an existential proposition in this same sense of ‘existential.’ Still, I also argue that Euclid does actually endorse the view that elementary constructions aim at establishing that some objects are available within EPG, to the effect that appropriate singular terms actually refer (and namely refer to such objects), and the corresponding concepts are actually instantiated (and are namely instantiated by such objects). This is, of course, perfectly compatible with (iv).



used in a broader sense, to label any geometrical object that has been or could be constructed (through an elementary construction) on the basis of some other given ones. I conform to this latter use, which is not only more liberal, but also very common in classical geometry.

Adopting this terminological convention is not enough, however, to answer the natural question that arises at this point. All the clauses of elementary constructions explicitly stated in the *Elements* license construction of some objects supposing that some other objects are given: Postulate I.1 licenses construction of a segment if two points are given; Postulate I.2 licenses construction of a given segment (that is, construction of two new segments if a segment is given); Postulate I.3 licenses construction of a circle if two points (or a segment) are given. But, if any construction requires that some objects are given beforehand, how can one construct unconditional objects in EPG? Take again the unconditional concept of equilateral triangle. How can one construct an equilateral triangle as such, that is, an equilateral triangle without any further specification relative to its relations with some given objects? And, if it were impossible to construct unconditional objects in EPG, how could one prove that they are available within EPG, and can then rightfully be taken as given and provide the starting point of an elementary construction of some other objects?

The answer goes in different stages.

The first consists in admitting that, among the clauses of elementary constructions, there is one (left implicit by Euclid) that licenses admitting, without any previous proof, that any (finite) number of arbitrary (i.e., not further specified) and thus unrelated segments is given. This means that an elementary construction can begin by supposing that any (finite) number of strokes representing these segments has been freely drawn.<sup>27</sup> Besides this, if it has been shown that, starting with such a stand, an object of a certain sort, or a certain system of related objects of some sorts can be constructed through an elementary construction, another such construction can then start (for short) with the admission that an object, a system of objects, or any (finite) number of arbitrary, and thus unrelated, such objects or systems of objects are given, and represented by appropriate freely drawn diagrams.<sup>28</sup>

If this is granted, it is enough to conventionally admit that taking something as the starting stage of an elementary construction entails having constructed it through such a construction, in order to infer that arbitrary segments can be constructed in EPG by force of this very clause, and this is enough to conclude that unconditional segments are available with EPG. According to Definition I.3, this also entails that unconditional points are so.<sup>29</sup>

<sup>27</sup> A particular case of this clause, involving only one arbitrary segment, is applied in the solution of Proposition I.1 expounded in footnotes 18 and 23, above. For another example, relative to an application of this same clause involving three arbitrary segments, cf. footnote 31, below. Also, the solution of Proposition I.2 that I shall mention later applies this same clause, in the case of two arbitrary segments. Because of Postulate I.1, one might alternatively admit an analogous clause in which arbitrary segments are replaced by arbitrary points. The reason that I prefer the former admission is that Definition I.3 suggests that segments have priority over points in elementary constructions, since it implies that, if a segment is obtained, two points are also *ipso facto* so, whereas a segment is not *ipso facto* obtained if two points are so.

<sup>28</sup> For example, Proposition I.42 of the *Elements* asks for the construction of a parallelogram equal to a given triangle and having an angle equal to a given one. This construction starts, then, with the admission that an arbitrary angle and an arbitrary triangle, unrelated to each other, are given and represented by appropriate freely drawn diagrams.

<sup>29</sup> Cf. footnote 27.



This being stated, the best way to explain how other unconditional objects can be constructed in EPG is through examples.

To begin with, take again Proposition I.1 of the *Elements*. It asks us, as said, to construct an equilateral triangle on a given segment.<sup>30</sup> At first glance, it then seems concerned with a conditional concept. But this is not so. Say, for short, that each EPG object falling under a concept *P* intrinsically includes one or more EPG objects falling under a concept *Q* if obtaining one or more objects falling under *Q* is an inescapable part of obtaining each object falling under *P*. Each triangle intrinsically includes three segments, for example. Hence, Proposition I.1 asks us to construct an object that intrinsically includes some segments one of which is taken to be given. As this last object is supposed to be arbitrary, the equilateral triangle that Proposition I.1 asks us to construct is specified only in virtue of the requirement that one of the objects that it intrinsically includes is an arbitrary given one. This seems to me to be enough for concluding that this equilateral triangle is arbitrary, in turn, to the effect that Proposition I.1 asks us to construct an arbitrary equilateral triangle, and is thus concerned with an unconditional concept.

It follows that the solution of this proposition shows that an arbitrary equilateral triangle can be constructed through an elementary construction, and it proves, thus, that (unconditional) equilateral triangles are available within EPG. This is the only way to prove that unconditional objects other than segments and points are available within EPG. In other terms, to prove that this is so, one has to prove that an arbitrary such object can be constructed through an elementary construction.

I term ‘ontological’ the function that an EPG problem complies with insofar as its solution proves that an arbitrary object falling under a certain unconditional concept can be constructed through an elementary construction, and, thus, that objects falling under this concept are available within EPG. Propositions I.22 and I.46 provide other examples, since they prove that generic triangles and squares are available within EPG.<sup>31</sup> For circles, things are much simpler, since Postulate I.3 allows constructing a circle having a given arbitrary point as center and radii equal to a given arbitrary segment. For angles, things are a little bit more complicated, but it is enough to admit that the point that in Proposition I.2 is taken to be given is the extremity of a segment for concluding that the solution of this proposition proves how to construct an angle that could be taken as arbitrary. By solving Propositions IV.11, IV.15, and IV.16, Euclid shows then, respectively, how to inscribe, through an elementary construction, a regular pentagon, a regular hexagon, and a regular pentadecagon into a given arbitrary circle, which immediately suggests how to construct these reg-

<sup>30</sup> On the solution of this proposition, cf. footnotes 18 and 23.

<sup>31</sup> Notice that the availability of generic triangles within EPG does not immediately follow from the implicit clause mentioned above about the starting point of an elementary construction. This clause licences, indeed, admitting, without any previous proof, that three arbitrary, and thus unrelated, segments are given and represented by three strokes freely drawn. But it does not license the same for three segments mutually placed to form a triangle. Hence, availability of generic triangles within EPG needs to be proved, and this is done in solving Proposition I.22. The fact that this proposition occurs so late in the *Elements*, and namely after generic triangles have already been considered in other propositions, could make one to think that it involves a sort of circularity. Still, this circularity is expository, at most, since the solution of this proposition relies only on Postulate I.3 and the solution of Proposition I.2 (and the restrictive condition involved in this same proposition could be avoided if the possibility of stating a problem with an impossible solution in some cases were admitted).

ular polygons on an arbitrary given segment,<sup>32</sup> and thus to prove that regular pentagons, hexagons, and pentadecagons are available within EPG.

Come back now to Proposition I.1. It is easy to see that the construction involved in its solution applies to any given segment (regardless of whether it is arbitrary or not). Hence, this solution also proves that for any given segment, an equilateral triangle (or better two, since, though Euclid does not remark on it explicitly, the same construction can be replicated on two opposite sides of the given segment) can be constructed on it through an elementary construction. This is a new constructive clause for elementary constructions derived from those already stated or implicitly admitted.<sup>33</sup> Let us term ‘constructive’ the functions that an EPG problem complies with insofar as its solution provides such a proof.

All EPG problems comply with this function, though this is negligible in many cases. In contrast, the great majority of EPG problems do not comply with the ontological one, since they are concerned with conditional concepts. Typically, EPG problems complying with the constructive function but not with the ontological one ask us to construct either conditional points, or other conditional objects that are *ipso facto* given or can easily be constructed through an elementary construction if such points are given.<sup>34</sup> Hence, their solutions prove that these conditional objects can be constructed through an elementary construction and are thus available within EPG.<sup>35</sup>

A last remark for completeness, before we turn our backs on EPG. The previous considerations also allow an explanation of the logical nature of theorems in EPG. These are, of course, universal statements about EPG objects. Still, insofar as these objects do not form a fixed domain of quantification, as explained above, it is not appropriate to understand them as claims that all EPG object of a certain sort, taken individually, enjoy a certain property. In my understanding, they are about objects available within EPG, and state that any given

<sup>32</sup> Suppose that a regular polygon has been inscribed into a given arbitrary circle. If an arbitrary segment is given, construct on this segment an isosceles triangle (having the two other sides equal to each other) similar to the isosceles triangle formed by a side of this polygon and two radii of this circle (which can be done without appealing to any proportion, merely by constructing appropriate parallel segments). The vertex of this triangle opposed to the given segment is the center of another circle into which can be inscribed a polygon similar to the given one and having the given segment as side.

<sup>33</sup> Suppose that, in conducting any argument, one is considering a certain segment. Then this new clause licenses constructing an equilateral triangle on it (or better two) having it as a side, in the same way as, for example, Postulate I.1 licenses constructing a segment joining two given points.

<sup>34</sup> Take Proposition I.9 as an example. It asks us to bisect a given angle, to the effect that its solution requires that a new angle be constructed within the given one, and, for this to be done, that an appropriate segment (or straight line) through the vertex of this last angle be constructed. Still, because of Postulate I.1, the construction of this segment immediately follows from the construction of an appropriate point (which Euclid identifies with the vertex of an equilateral triangle constructed, according to the solution of Proposition I.1, on a chord of the given angle). Another example is given by Proposition I.10, which asks us, instead, to bisect a given segment. Its solution requires that a new segment be constructed on the given one. But it is quite clear that such a new segment is *ipso facto* given if an appropriate point is constructed on the segment originally given.

<sup>35</sup> The proof that a conditional object can be constructed through an elementary construction starting from the relevant given objects could be also seen as the proof that a certain arbitrary configurations of objects can be so constructed. This does not undermine the distinction between the ontological and the constructive functions of EPG problems, however, since the fact that some configurations of geometrical objects are taken as genuine geometrical objects and some are not is crucial. And it is just on this fact that this difference rests, in the very end.

EPG object of a certain sort enjoys a certain property, or better that, if a given EPG object is of a certain sort, then it enjoys this property. For example, Proposition I.5 of the *Elements* does not state, in my view, that all isosceles triangles have the angles at the base equal to one another, but rather that if an isosceles triangle is given, then its angles at the base are equal to one another (which is perfectly reflected by its proof which, as is well known, concerns an arbitrary given isosceles triangle).

### 2.3. Extending Euclid's plane geometry before Descartes

Insofar as the only way to prove that EPG objects other than arbitrary segments and points are available within EPG is by solving problems, there is no general warrant that EPG objects are available within EPG. And, as a matter of fact, many of them, both unconditional and conditional, are not. For example, both regular heptagons and squares equal to given circles are EPG objects (respectively unconditional and conditional), but are not available within EPG.<sup>36</sup> Informally speaking, one can say that conservatively extending EPG means accepting EPG and just licensing other ways to obtain geometrical objects than by elementary constructions (which implies that also in the extensions of EPG that are gotten this way, geometrical objects are not taken to exist, in the sense explained above, but are just required to be obtained in some appropriate way). In this sense, the search for constructions of EPG objects not available within EPG was a search for a conservative extension of EPG. Still, some of these constructions also involved geometrical non-EPG objects (i.e., objects that EPG does not take into account, or that are purported to fall under concepts whose application conditions cannot be stated in the language of EPG). The desire to study these objects was, also, a motivation for looking for such an extension. In Section 3, I shall account for the way Descartes pursues this aim. Before that, it is useful to consider some earlier and more local efforts for solving geometrical problems through various sorts of nonelementary constructions.

These constructions were still diagrammatic, to the effect that the identity conditions of the objects obtained though they were also provided by appropriate diagrams. Hence, these objects no longer formed a fixed domain of quantification and individual reference, in the sense explained in Section 2.1, and to refer individually to some of them, one had to obtain, or suppose to have obtained them by drawing, or imagining to draw appropriate diagrams. Still, some of these objects were defined by describing their construction, to the effect that their definition already provided a warrant for their availability.

The admission of nonelementary constructions also made it possible to state two sorts of non-EPG problems. The former includes problems that, like EPG ones, ask us to construct EPG objects, but, unlike them, do not require this to be done through an elementary construction. I call them 'quasi-EPG problems.' The latter includes problems asking for the construction of some non-EPG objects, such as curves other than circles (such as in Pappus's problem [Pappus [CMH], II, 676–683; Pappus [C7SJ], I, 118–123]), or conditional EPG objects that are required to stay in some appropriate relations with some given non-EPG objects (such as in the problem of tangents to conics). I call them 'strictly non-EPG problems.'

<sup>36</sup> The claim that some EPG objects are not available within EPG could appear odd. However, it seems to me a quite natural way of rendering the hiatus that there is in EPG between definitions and constructions, which is often rendered (wrongly I think) by saying that defining objects in EPG is not enough for warranting for their existence.

For my present purpose, I only need concentrate on quasi-EPG problems. Any EPG problem is of course easily convertible to a quasi-EPG problem by merely omitting the requirement that the objects be constructed through an elementary construction. Relevant quasi-EPG problems, however, include only those that either are nonsolvable through an elementary construction, or at least withstood a solution through an elementary construction in classical geometry though actually admitting such a solution (a well-known example is the problem of constructing a regular heptadecagon on a given segment, which was famously proved to be solvable through an elementary construction only by Gauss: [Zimmermann 1796; Gauss 1801, Sect. VII]).

I also suggest distinguishing two sorts of nonelementary constructions that, in classical geometry, entered into the solution of those problems. The former rely only on EPG objects, though applying constructive clauses not included among those of elementary constructions; the latter rely instead on some non-EPG objects, namely curves other than circles. Hence, whereas the former include only constructions of EPG objects, the latter include constructions of such curves. I call the former ‘quasi-elementary constructions’ and the latter ‘strictly nonelementary constructions.’ Quasi-EPG problems are possibly solvable either through the former or through the latter. Strictly non-EPG ones are possibly solvable, instead, only through the latter.

Admitting quasi-elementary constructions is the same as extending EPG exactness norms. An aspect of the exactness concern for early modern geometry was relative to such a sort of extension, and was thus specifically related to quasi-elementary constructions. Another aspect of this concern was specifically related, instead, to strictly nonelementary constructions and was relative to the admission of appropriate exactness norms relative to curves other than circles.

The distinction between quasi-elementary and strictly nonelementary constructions is still not enough to account for the variety of constructions that populated classical geometry. For each one of these two sorts of constructions, finer distinctions are possible and necessary. For many problems of classical geometry, especially for quasi-EPG ones, different solutions involving different sorts of constructions were known, indeed. The preference for one of them over others, or the search for new solutions, essentially different from those already known, were symptoms of different attitudes towards the exactness concern. In the first part of the 17th century, many of these attitudes cohabited, being often only locally motivated, or not motivated at all. When Descartes came to geometry, he faced such a plurality of attitudes. Hence, his views on this topic could and should be understood as reactions to this quite confused state of affairs.

It is not easy, however, to make the relevant distinctions clear without going through appropriate examples. The purpose of the following Sections 2.3.1, 2.3.2, and 2.3.3 is to offer these examples by considering different solutions of three classical quasi-EPG problems: the angle trisection, the two mean proportionals, and the circle-squaring ones.<sup>37</sup>

### 2.3.1. *Trisecting an angle*

Let us begin with Viète’s solution of the first of these problems [Viète 1593, Prop. IX; Viète [AAKW], 398; Bos 2001, 167–173]. Let  $\widehat{EBD}$  (Fig. 1) be the angle to be trisected (Viète’s construction applies regardless of whether it is acute, right, or obtuse). Trace the

<sup>37</sup> These are also the problems Serfati refers to in order to describe the historical context of Descartes’s reflections on geometrical exactness: Serfati [1993, 198–204].

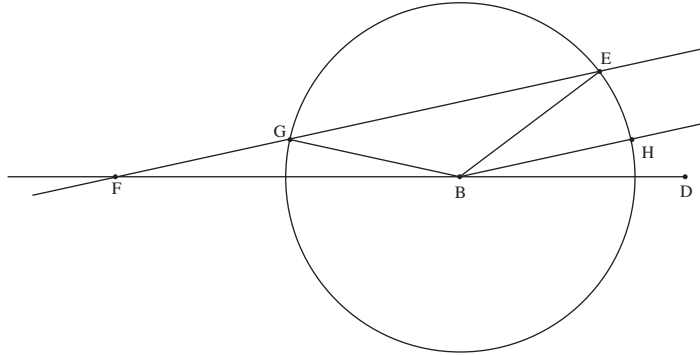


Figure 1. Viète's solution of the angle trisection problem.

circle with radius  $BE$  and center  $B$ , and the straight line  $FE$  such that  $FG = BE$ . If  $BH$  is the parallel to  $FE$  through  $B$ , then  $\widehat{HBD}$  is the third part of  $\widehat{EBD}$ . The proof is easy by considering the internal angles of the isosceles triangles  $FBG$  and  $GBE$ .

The straight line  $FE$ , i.e., the points  $G$  or  $F$ , cannot be obtained through an elementary construction. To license their construction, Viète appeals to a new “postulate,” apt “to supply the deficiency of geometry.” This is the *neusis* postulate, which licenses one “to draw a straight line from any point to any two [given] lines, so that the intercept between them be any possible determined segment” (I slightly modify Bos’s translation [Bos 2001, 168]; this significantly differs in turn from Witmer’s [Viète [AAKW], 388]).<sup>38</sup> The previous construction applies this postulate in the case where one of the two given lines is straight and the other is a circle.

This construction results from a slight modification of that offered by Pappus in his *Mathematical Collection* [Pappus [CMH], I, 271–277; Heath 1961, I, 235–237; Knorr 1989, 213–216; Bos 2001, 53–56], which requires, instead, cutting off a segment equal to a given one between two straight lines.<sup>39</sup> Pappus does not appeal to any new postulate,

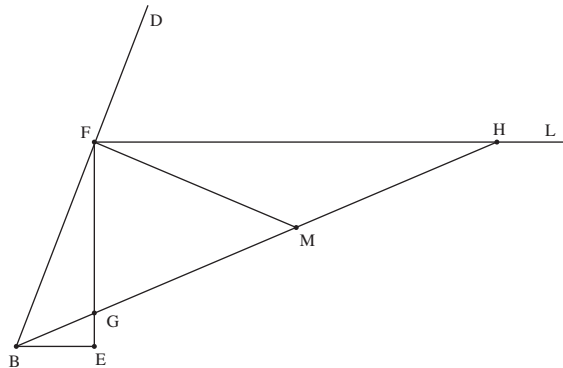
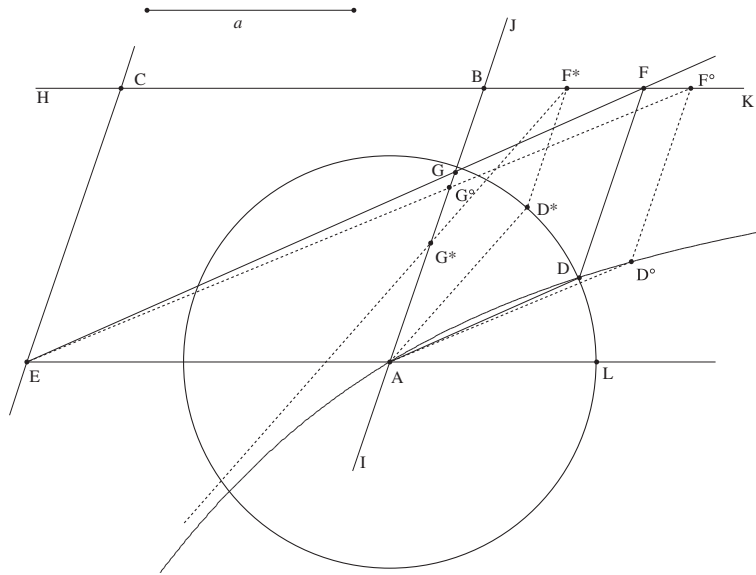


Figure 2. Pappus's solution of the angle trisection problem.

<sup>38</sup> For the meaning of the adjective ‘possible [*possibili*],’ cf. Bos [2001, 168, footnote 4].

<sup>39</sup> Let  $\widehat{DBE}$  (Fig. 2) be the angle to be trisected (which is assumed to be acute; if the given angle is obtuse, an analogous construction provides a trisection of its supplement). Trace  $EF$  and  $FL$  respectively perpendicular and parallel to  $BE$ . Trace  $BH$  such that  $GH = 2BF$ .  $\widehat{HBE}$  is the third part of  $\widehat{DBE}$ . Supposing that  $M$  is the middle point of  $GH$ , the proof involves the isosceles triangles  $HFM$  and  $FBM$ .

Figure 3. Pappus's *neusis* construction.

however. He rather shows how to obtain a *neusis* by intersection of a circle and a hyperbola. Let it be requested to trace a straight line through E (Fig. 3) cutting off a segment equal to  $a$  between HK and IJ. Let EA and EC be respectively the parallels to HK and IJ through E. Trace the circle of center A and radius AL equal to  $a$ , and the hyperbola through A with asymptotes HK and EC. Let D be the intersection point of this circle and this hyperbola. The parallel EG to AD through E is such that  $GF = a$ . To prove it, it is enough to remark that any point  $D^*$  on the circle is such that  $G^*F^* = a$ , if  $D^*F^*$  and  $G^*F^*$  are respectively parallel to IJ and  $AD^*$ , and any point  $D^\circ$  on the hyperbola is such that  $G^\circ F^\circ$  passes through E, if  $D^\circ F^\circ$  and  $G^\circ F^\circ$  are respectively parallel to IJ and  $AD^\circ$ .

A hyperbola is univocally determined if its asymptotes and a point through which it passes are given. Hence, Pappus's solution requires admitting that a hyperbola is *ipso facto* obtained—so that intersection points of it and some other given lines are obtained in turn—if it is univocally determined. In classical geometry, this was usual for any conic. Commandinus's Latin translation of Pappus's *Collection* [Pappus [CMC]] appeared five years before Viète published his solution. It is thus likely that Viète preferred constructions based on his new postulate over ones depending on this admission with a full knowledge of the facts. It is not easy to say why, but some guesses are possible.

To justify his postulate, Viète remarks that Nicomedes “seems to have performed” constructions that Viète's postulate would license, by relying on appropriate conchoids of straight lines and circles.<sup>40</sup> Conchoids of straight lines are defined by Pappus [Pappus [CMH], I, 242–245] as trajectories of a point P (Fig. 4.1) moving on a straight half-line OP so that MP remains constant while this line rotates around a fixed point O cutting a

<sup>40</sup> Viète also appeals to Archimedes, who, in proving Propositions 5–9 of his treatise *On spirals*, admits constructions based on *neusis* [Heath 1897, C–CXXII].



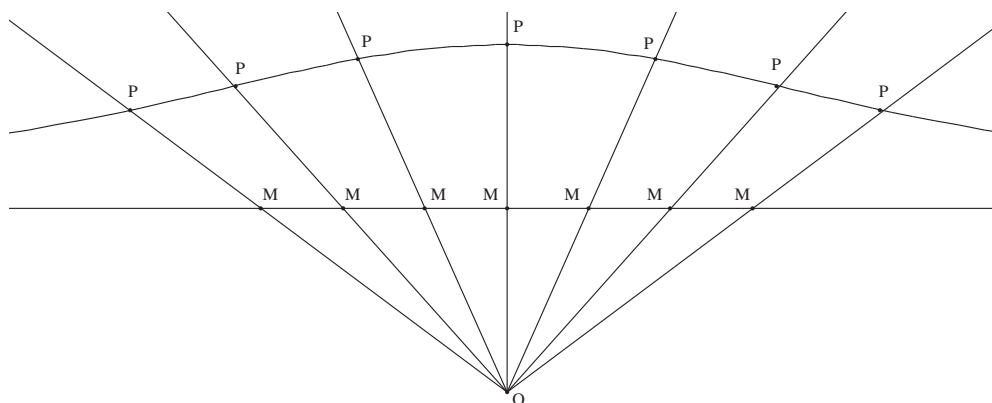


Figure 4.1. External branch of a conchoid of a straight line.

fixed straight line  $MM$ .<sup>41</sup> This foreshadows the possibility of tracing this curve through a simple instrument composed of three rulers replacing  $OP$ ,  $MM$ , and  $MP$ . An analogous instrument can be used to trace conchoids of circles. For future reference, call these instruments ‘conchoid compasses.’

Like regular compasses and other similar instruments, they can be used in two ways: either in the tracing way, i.e., by making them trace a curve; or in the pointing way, i.e., by making them indicate some points (which are then taken to be obtained) under the condition that some of their elements coincide with some given geometrical objects, or meet some other conditions relative to given objects.<sup>42</sup> If an instrument is used in the former way, once a curve is traced, it can be put away, and this curve taken as constructed. If it is used in the latter way, the sought-after points can only be indicated by appropriate elements of it. To construct these points, one has to transpose this indication on the support where the construction is done through appropriate dots or other diagrammatic marks. This suggests two different sorts of constructive clauses, licensing respectively obtaining curves by tracing them through instruments, and obtaining points by using instruments in the pointing way. The former necessarily pertains to strictly nonelementary constructions; the latter can pertain either to strictly nonelementary constructions or to quasi-elementary ones.

<sup>41</sup> This is only the external branch of a conchoid, in fact. To have the entire conchoid, including an external and an internal branch (Fig. 4.2), one has to take the point  $P$  to be on the entire straight line  $OP$  and, so to say, to change sides with respect to  $O$  and  $M$  while passing through the infinite. Alternatively, one can take the entire conchoid to be formed by the trajectories of two points  $P$  and  $Q$  placed on the straight half-line  $OP$  at equal distance from  $M$ , on the two sides of it. To have a conchoid of a circle, it is enough to replace the straight line  $MM$  with a circle (Fig. 4.3). For short, in what follows, I shall use the term ‘conchoid’ to refer only to the external branch of a conchoid of a straight line.

<sup>42</sup> To illustrate the use of conchoid compasses in the pointing way, consider Pappus’s construction related in footnote 39, above. After having traced  $EF$  and  $FL$  (Fig. 5.1), one can make use of a compass whose pole  $O$  is brought to coincide with the vertex  $B$  of the angle  $DBE$  and whose fixed rule  $HK$  coincides with  $EF$ . If the ruler  $XY$ , sliding on the other ruler  $OW$ , is equal to the double of  $BF$ , it is enough to rotate  $OW$  around  $O$  until  $Y$  comes to be on  $FL$ , for  $OW$  be in the position that the segment  $BH$  has to take for  $GH$  to be equal to  $2BF$ , as required. The same compass is used in the tracing way if it is so used that, while  $OW$  (Fig. 5.2) rotates around  $O$ ,  $Y$  traces the conchoid  $IJ$  whose intersection with  $FL$  provides the point  $H$ .

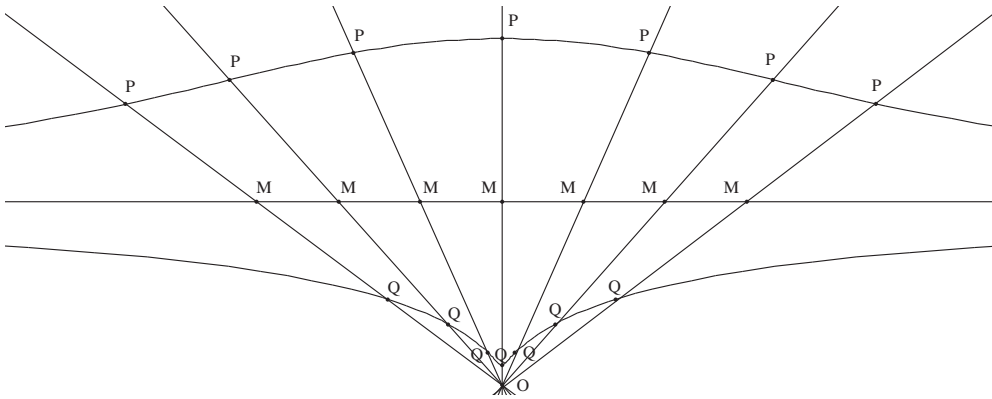


Figure 4.2. Conchoid of a straight line.

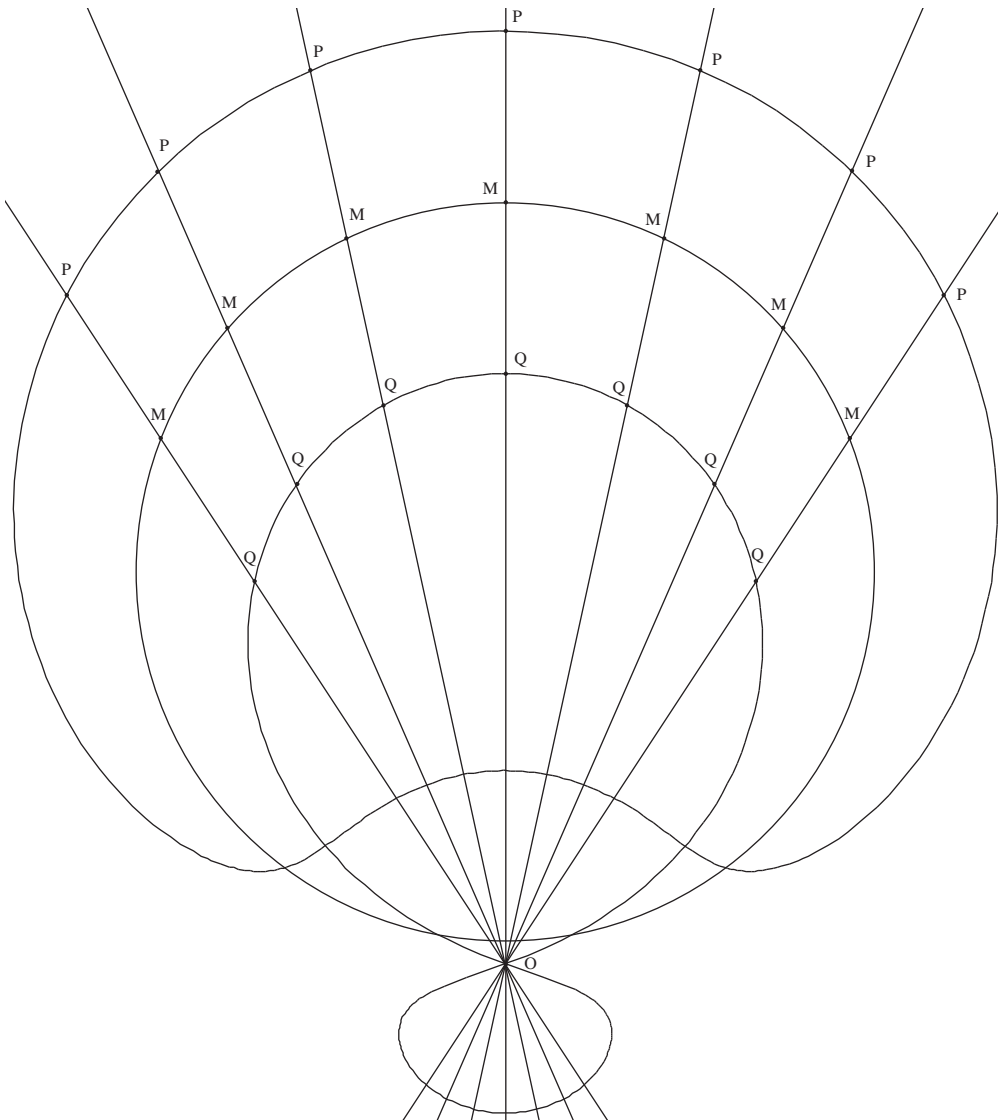


Figure 4.3. Conchoid of a circle.



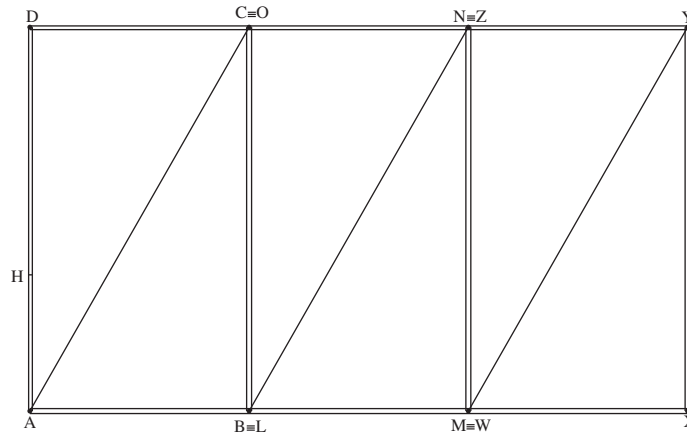


Figure 6.1. Eratosthenes's solution of the two mean proportionals problem, stage 1.

tary construction involving the intersection of conics, and on the admission that a conic is *ipso facto* obtained if it is univocally determined.

The former goes as follows. Let  $a$  and  $b$  be two given segments ( $a < b$ ). Let also ABCD, LMNO, and WXYZ (Fig. 6.1) be three equal rectangular plates of height equal to  $b$  (their length is irrelevant). On each of them trace a diagonal, and mark on AD a length AH equal to  $a$ . Slide LMNO beneath WXYZ (Fig. 6.2) so that LN cuts WZ in some point Q. Trace QY and produce it up to cut AC in K, AD in H', and AX produced in E. Slide ABCD beneath LMNO (Fig. 6.3) until K falls on LO. If H' coincides with H, stop the procedure. Otherwise, slide LMNO and AMCD (Fig. 6.4) again until this happens. The two mean proportionals between  $a$  and  $b$  are equal to LK and WQ, respectively. The proof is obvious by similarity of triangles.

One could imagine replacing the plates with genuine geometrical rectangles. But this would make no relevant difference, since the diagrams representing these rectangles would then be assumed to move until they reach a position that satisfies a coincidence condition relative to other diagrams representing some given geometrical objects. This would be a use of diagrams essentially different from that involved in elementary or in other sorts of non-elementary constructions, where coincidences are not acknowledged by inspecting moving diagrams but imposed on fixed diagrams by drawing them.<sup>43</sup>

A simplification of Eratosthenes's construction was suggested by Clavius [Clavius 1589, 33; Bos 2001, 72–75]. Let AB (Fig. 8.1) be equal to  $b$ , and C be a point on it such that

<sup>43</sup> Eratosthenes's instrument could also be used in the tracing way, but the curves traced by it should then be supposed to move until they reach a position that meets a coincidence condition. The resulting construction is thus more complicated than both Menaechmus's and Eratosthenes's and presents no advantages over them. This is how one could reason. While the plates slide, the intersection point  $G^*$  of AC and HY (Fig. 7) describes an arc of a hyperbola which is fixed under the variation of the distance LW. If this distance varies while AL remains fixed, the intersection point  $G^\circ$  of AC and YQ describes another arc of a hyperbola whose position depends on AL. Let  $K^*$  be the intersection point of these two hyperbolas. Trace  $K^*Y$  intersecting WZ at  $Q^*$ . Trace  $Q^*L^*$  parallel to AC, and  $L^*K^\circ$  perpendicular to AX. The positions of points K and  $K^\circ$  vary with AL. When they come to coincide, Q and  $Q^*$  do also. This is the final configuration: the sought-after mean proportionals are equal to LK and WQ, respectively.

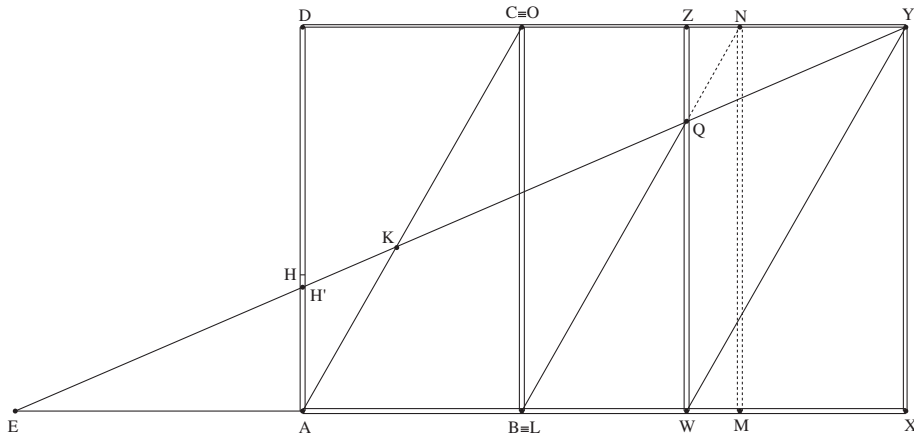


Figure 6.2. Eratosthenes's solution of the two mean proportionals problem, stage 2.

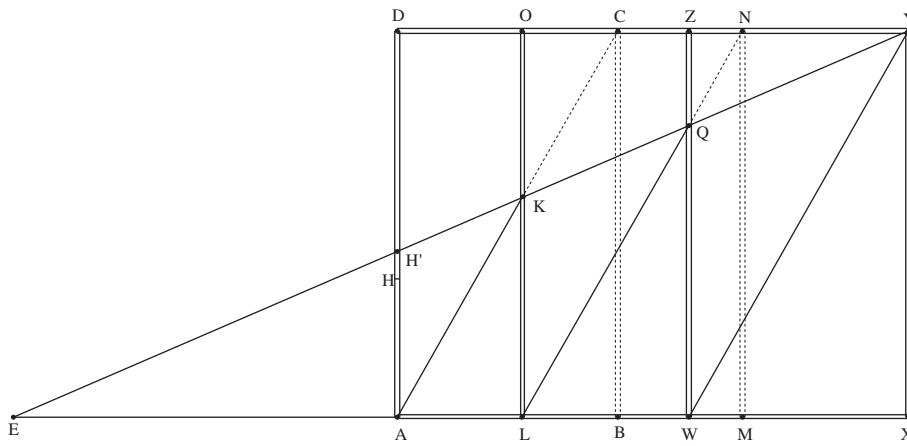


Figure 6.3. Eratosthenes's solution of the two mean proportionals problem, stage 3.

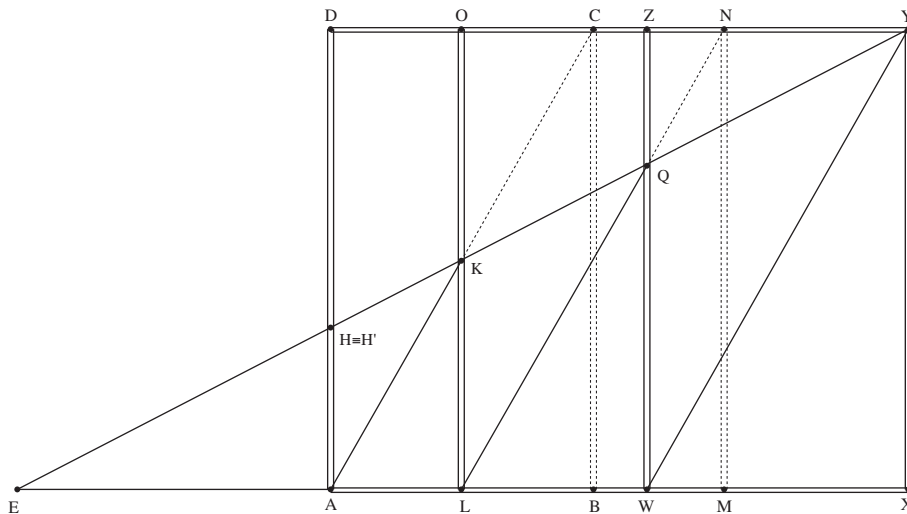


Figure 6.4. Eratosthenes's solution of the two mean proportionals problem, stage 4.

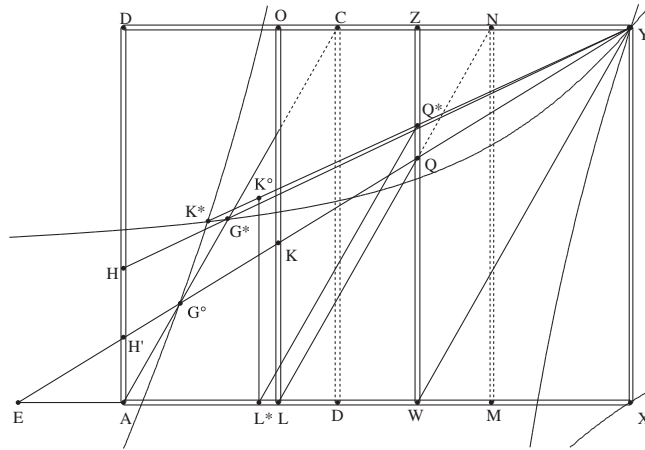


Figure 7. Using Eratosthenes's instrument in the tracing way.

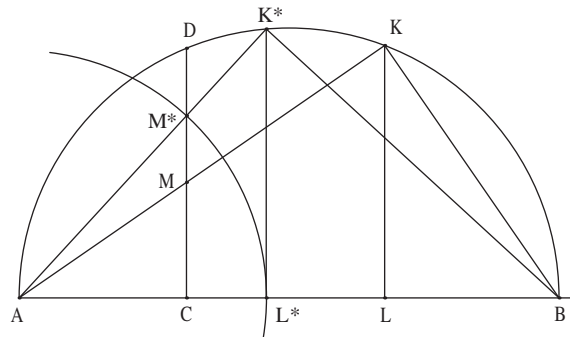


Figure 8.1. Clavius's solution of the two mean proportionals problem.

$AC = a$ . Trace the semicircle of diameter  $AB$  and its chord  $AK$  through any point  $K$  on it. Let  $CD$  and  $LK$  both be perpendicular to  $AB$ , and  $M$  be the intersection point of  $CD$  and  $AK$  (possibly extended). Let  $K^*$ ,  $L^*$ , and  $M^*$  be the respective positions of  $K$ ,  $L$ , and  $M$  such that  $AM^* = AL^*$ . The sought-after mean proportionals are equal to  $AM^*$  and  $AK^*$ . The proof is immediate by the similarity of triangles  $ACM^*$ ,  $AL^*K^*$ , and  $ABK^*$ .

Clavius admits that points  $K^*$ ,  $L^*$ , and  $M^*$  are obtained but without saying how. They could be obtained by using an appropriate instrument in the pointing way. Let  $AP$  and  $AQ$  (Fig. 8.2) be two rulers, the former of which is fixed and the latter of which rotates around  $A$ . Attach to them three other rulers,  $CD$  and  $LK$ , both perpendicular to  $AP$ , and  $BK$ , perpendicular to  $AQ$ , so that:  $C$  is fixed on both  $AP$  and  $CD$ ;  $M$  slides on both  $CD$  and  $AQ$ ;  $L$  is fixed on  $LK$  and slides on  $AP$ ;  $K$  is fixed on  $BK$  and slides on both  $LK$  and  $AQ$ ; and  $B$  slides on both  $AP$  and  $BK$ . If  $AC = a$ , and the instrument is so adjusted that  $AB = b$  and  $AM = AL$ , the sought-after mean proportionals are equal to  $AM$  and  $AK$ . To ensure that  $AC = a$  and  $AB = b$ , it is enough to evaluate whether  $A$ ,  $B$ , and  $C$  coincide with the extremities of two given segments. But  $AM$  and  $AL$  cannot be superposed, and the only way to construct two equal segments that they have to coincide with is by solving the two mean proportionals problem itself. Hence, to ensure that  $AM = AL$ , while using this instrument in solving the problem, one has to measure them, for example, by graduating  $AP$  and  $AQ$ , or by equipping the instrument with a graduated disc centered on  $A$ . Using such an





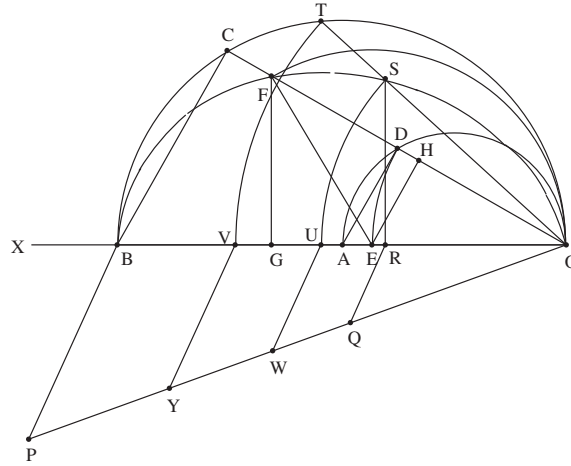


Figure 9.1. Villalpando's solution of the two mean proportionals problem.

generated by C moving on the semicircle BCO. Let  $PO = b$  form any angle with BO and let Q be taken on it, R on BO, and S on BFO so that  $QO = a$ , QR is parallel to PB, and RS is perpendicular to BO. Through O and S, trace the chord OT. Take U and V on BO, and W and Y on PO so that  $UO = OS$ ,  $VO = OT$ , and WU and YV are parallels to PB. The sought-after mean proportionals are equal to WO and YO. To prove it, trace EH perpendicular to OC and remark that  $OC = 2OD$  and  $OF = 2OH$ , to the effect that  $OH : EO = OF : OC$ , and  $GO : OF = OF : OC = OC : BO$ . As S is on the locus, also the proportions  $RO : OS = OS : OT = OT : BO$  hold, and UO and VO are thus two mean proportionals between RO and BO.

The problem of finding two mean proportionals between two given segments  $a$  and  $b$  can be reduced to that of finding two mean proportionals between another given segment  $\beta$  and the fourth proportional  $\alpha$  between  $b$ ,  $a$ , and  $\beta$ . Indeed, if  $\xi$  and  $\kappa$  are two mean proportionals between  $\alpha$  and  $\beta$  and  $\alpha : a = \xi : x = \kappa : y$ , then  $x$  and  $y$  are two mean proportionals between  $a$  and  $b$ . By exploiting this fact, Villalpando shows how to solve the problem by relying on a locus like BFO relative to any arbitrary given semicircle, whatever the two given segments  $a$  and  $b$  might be.<sup>44</sup> It is the locus of a point (F) obtained through an elementary construction based on the supposition that its generating point (C) is given in an arbitrary position. Villalpando's solution can thus be accepted only if it is admitted that a locus of a point is *ipso facto* obtained if this point is obtained through an admissible construction based on the supposition that the generating point of this same locus is given in an arbitrary position. This is the paradigmatic case of what Bos calls 'generic point-wise constructions' [Bos 2001, 343].

But Villalpando's locus can also be traced using an appropriate instrument. Let IJ (Fig. 9.2) be a ruler longer than  $2b$ , O its middle point, and B another point on it such that  $BO = b$ . Let IM and JM be two rulers forming a right angle at M. Let OM and BC be two other rulers also forming a right angle at their intersection point C, and LN and LF be two further rulers, the former perpendicular to JM and passing through C, and the latter

<sup>44</sup> This locus is a branch of the sextic curve with equation  $(x^2 + y^2)^3 = b^2 x^4$ , with respect to orthogonal coordinates of origin O and axis OX, with  $b = BO$  [Bos 2001, 76; footnote 33]. The whole sextic forms a figure of eight including four similar symmetric branches with a common vertical tangent at O.



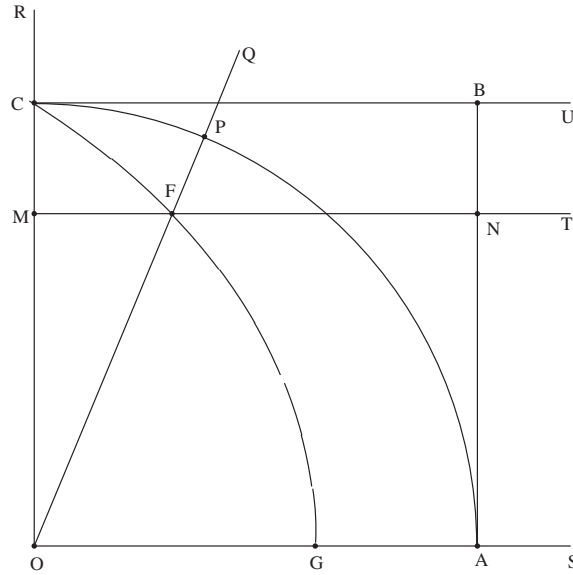


Figure 10.1. Quadratrix as defined by Pappus.

A way to construct any number of specific points of a quadratrix through an elementary construction is suggested by Clavius [Clavius 1589, I, 894–918, esp. 896–896; Mancosu 1996, 74–76; Bos 2001, 160–166]. One begins by constructing the intersection point  $F_{1,1}$  (Fig. 10.2) of the bisector  $OF_{1,1}$  of the right angle  $\widehat{COA}$  and the perpendicular  $M_{1,1}F_{1,1}$  to  $OC$  through its middle point  $M_{1,1}$ . Then one continues in the same way, by constructing the intersection points  $F_{2,1}$  and  $F_{2,2}$  of the bisectors  $OF_{2,1}$  and  $OF_{2,2}$  of the angles  $\widehat{COF_{1,1}}$  and  $\widehat{F_{1,1}OA}$  and the perpendiculars  $M_{2,1}F_{2,1}$  and  $M_{2,2}F_{2,2}$  to  $OC$  through the middle points  $M_{2,1}$  and  $M_{2,2}$  of  $M_{1,1}C$  and  $OM_{1,1}$ , respectively. By reiterating this construction, one constructs as many points  $F_{1,1}$ ,  $F_{2,1}$ ,  $F_{2,2}$ ,  $F_{3,1}$ ,  $F_{3,2}$ ,  $F_{3,3}$ ,  $F_{3,4}$ , etc. as one wants, all belonging to the quadratrix.

Though these points are specific, for Clavius their construction provides a “geometrical description” of the corresponding quadratrix that is enough, according to him, for obtaining such a curve if  $OC$  is given. If this is admitted, or it is admitted that a quadratrix is obtained in some way, then it is easy to rely on it for dividing any given angle according to any rational ratio [Pappus [CMH], I, 284–287; Knorr 1986, 84; Bos 2001, 43–44]. Since  $\widehat{POA}$  (Fig. 10.1) is to a right angle as  $OM$  is to  $OC$ , for any ratio  $\rho$  smaller than 1, any given angle  $RST$  (Fig. 10.3), and any quadratrix  $UVX$  of horizontal axis  $ST$ , if  $WV$  is the perpendicular to  $ST$  through the intersection point  $V$  of this quadratrix and the side  $RS$  of this angle, it is cut at  $Y$  so that  $WY$  is to  $WV$  in the ratio  $\rho$ , and  $YZ$  is parallel to  $ST$ , then  $\widehat{ZST}$  is to  $\widehat{RST}$  in this same ratio.

But quadratrices also famously have another property: the point  $G$  at which they cut their horizontal axis (Fig. 10.1) is such that  $OC$  is a mean proportional between the arc  $CPA$  and  $OG$ . This is proved by Pappus by *reductio ad absurdum* [Pappus [CMH], I, 256–259], and makes it possible to rely on a quadratrix to solve the circle-squaring problem through a strictly nonelementary construction. Let  $CPA$  (Fig. 10.4) be any given arc of a quarter circle, and  $CFG$  the corresponding quadratrix. If  $GC$  and  $CH$  are mutually perpendicular,  $OH$  is equal to the arc  $CPA$ . Hence, if  $B$  is the middle point of  $OC$ , the rectangle  $HOBK$  is equal to the quarter circle  $CPAO$ .

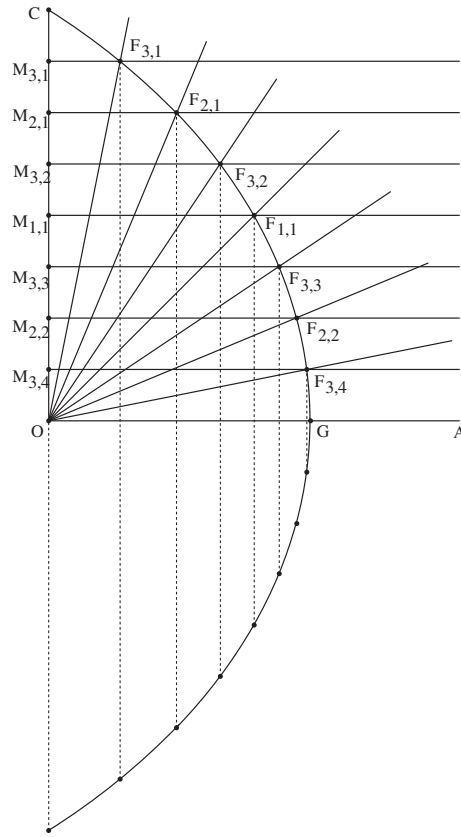


Figure 10.2. Clavius's construction of any number of specific points of a quadratrix.

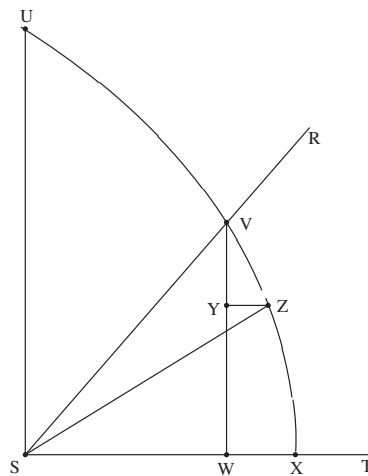


Figure 10.3. Dividing an angle according to a rational ratio.

If the quadratrix is assumed to be obtained by tracing it as Pappus suggests, this solution is open to Sporus's objections [Pappus [CMH], I, 252–257]. They are two. The first makes a statement of circularity: the constant speeds of the motions of OP and MN (Fig. 10.1)





appropriate such instruments. Others again involve curves obtained through generic point-wise constructions. Finally, some involve curves obtained by interpolation.

### 3. Descartes's exactness

EPG is often described as dealing with ideal and immutable self-standing objects or forms, which we can only inaccurately depict.<sup>47</sup> If EPG were so understood, the use of instruments in geometry (both in the pointing and in the tracing way), and more generally the appeal to motion, should be considered as entirely extraneous to its spirit, unless they were merely seen as tricks for achieving convenient depictions of ideal forms. The situation is different if it is granted that EPG objects are obtained through diagrammatic constructions. It then becomes natural to consider the admission of new procedures for drawing diagrams, also by using instruments, as a proper way of conservatively extending EPG.

In classical geometry, the use of instruments to obtain geometrical objects did not go together with fixing precise conditions that such a use of an instrument had to submit to. As a matter of fact, this made the exactness norms of geometric objects inaccurate and contributed highly to the fluidity of classical geometry. More generally, this fluidity depended on the fact that different sorts of nonelementary constructions were either admitted or rejected by appealing to different sorts of arguments, or even without relying on any precise argument.

Different opinions have been advanced on the evolution of Descartes's views on geometry: some insisting on the occurrence of essential changes, other on a substantial continuity of thinking. I cannot discuss these opinions here. I merely advance that, at least on one basic point, Descartes's views did not change from his youth until the *Géométrie*. He always aimed to overcome a situation like that just described, by imposing some global principles motivating an all-embracing attitude concerning constructions. A detailed account of the different ways in which he pursued this aim in different periods of his life is outside the scope of the present paper. I shall limit myself to the *Géométrie*, by showing that these principles, though based on a general standard of conceivability, remained faithful in spirit to EPG restrictions.

#### 3.1. Descartes on the mean proportionals problem

A convenient way to approach the matter is by considering what Descartes says about the mean proportionals problems at the beginning of the third book [Descartes 1637, 369–371; Descartes [AT], VI, 442–444; Bos 2001, 239–242]. As he famously refers to an instrument introduced in the second book [Descartes 1637, 317–319; Descartes [AT], VI, 391–392], which I have already mentioned in Section 2.3.2, some remarks on this instrument are necessary first. It is usually called ‘mesolabum’ or ‘proportions compass.’ I prefer the latter name, since the former also sometimes denotes Eratosthenes's instrument described in this same section or others inspired by it [Bos 2001, 35–36, 48, 72].

<sup>47</sup> This account is often taken to be a Platonic one. Still, though inspired by Proclus's neo-Platonic interpretation of EPG [Proclus [CITF]; Proclus [CIPM]], it contrasts with some recent understandings of Plato's conception of geometry, such as Burnyeat's [1987].

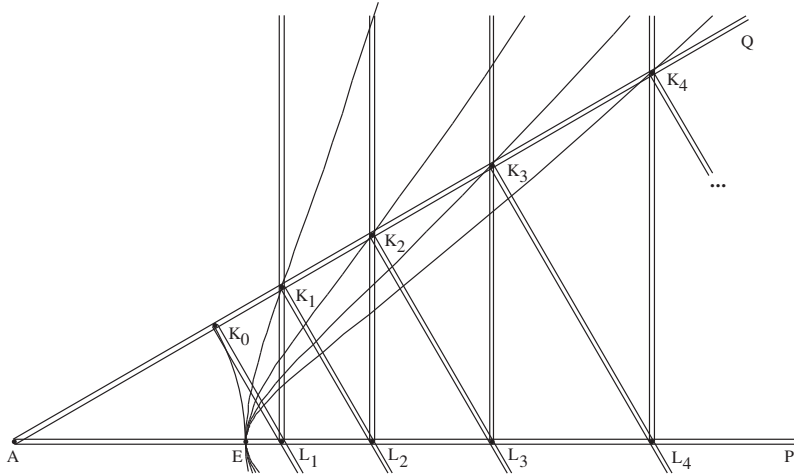


Figure 11.1. Proportions compass.

Consider the instrument represented in Fig. 8.3. For further convenience, change the names of points L, K and B and call them ‘ $L_1$ ,’ ‘ $K_1$ ,’ and ‘ $L_2$ ,’ respectively (Fig. 11.1). Then complete it by extending the pattern  $K_0L_1K_1L_2$  beyond  $L_2$ : attach to  $L_i$  ( $i = 2, 3, \dots$ ) the rulers  $L_iK_i$  perpendicular to AP, and to  $K_i$  the rulers  $L_{i+1}K_i$  perpendicular to AQ. Keep  $K_0$  fixed on both AQ and  $L_1K_0$ , while the points  $L_i$  ( $i = 1, 2, \dots$ ) slide on the rulers AP and  $L_iK_{i-1}$  and remain fixed on the rulers  $L_iK_i$ , and the points  $K_i$  slide on the rulers AQ and  $L_iK_i$  and remain fixed on the rulers  $L_{i+1}K_i$ . The triangles  $K_{i-1}AL_i$  and  $K_iAL_i$  are similar to each other. Hence  $AK_{i-1} : AL_i = AL_i : AK_i = AK_i : AL_{i+1}$ . If  $AK_0 = a$ , it follows that: (i) if the compass is so arranged that  $AK_i = b$ , then segments respectively equal to  $AL_1, AK_1, AL_2, \dots, AK_{i-1}, AL_i$  are  $2i - 1$  mean proportionals between  $a$  and  $b$ ; (ii) if the compass is so arranged that  $AL_{i+1} = b$ , then segments respectively equal to  $AL_1, AK_1, AL_2, \dots, AL_i, AK_i$  are  $2i$  mean proportionals between  $a$  and  $b$ .

This makes it obvious how to use proportions compasses in the pointing way to solve mean proportionals problems. Still, Descartes does not suggest using them this way. He rather shows how to rely on the curves  $EK_i$  traced by points  $K_i$  while AQ rotates around A, for constructing any even number of mean proportionals. Let it be required to construct  $2\mu$  mean proportionals between  $a$  and  $b$  (for some positive integer  $\mu$ , and supposing that  $a < b$ ). Descartes’s construction goes as follows. Let  $ae$  (Fig. 11.2) be equal to  $a$ . Produce it up to  $l'$  so that  $al' = b$ . Trace the circle with diameter  $al'$ , and apply to  $e$  the curve traced by the point  $K_\mu$  of a proportions compass with  $AK_0 = a$ , in such a way that  $e$  coincides with its origin marked by point E on the compass. Let  $k'$  be the intersection point of this circle and this curve. Join  $a$  and  $k'$  and, with center  $a$ , trace the circle of radius  $ae$  cutting  $ak'$  at  $k$ . Trace  $lk$  perpendicular to  $ak'$ . Then,  $al$  and  $ak'$  are respectively equal to the smallest and the greatest of the  $2\mu$  sought-after mean proportionals. If  $\mu > 1$ , the other  $2\mu - 2$  can be constructed as fourth proportionals.

Though Descartes does not note it, an analogous construction allows the construction of any odd number of mean proportionals. It goes as follows. Let it be required to find  $2\mu - 1$  mean proportionals between two given segments  $a$  and  $b$  (for some integer  $\mu$  greater than 1, and supposing that  $a < b$ ). Assuming that  $ae$  and  $ag$  (Fig. 11.3) are two segments with a common extremity that are respectively equal to  $a$  and  $b$ , one can proceed as follows (regardless of the angle  $\widehat{gae}$ ). Apply to  $e$  the curve traced by the point  $K_\mu$  of a proportions

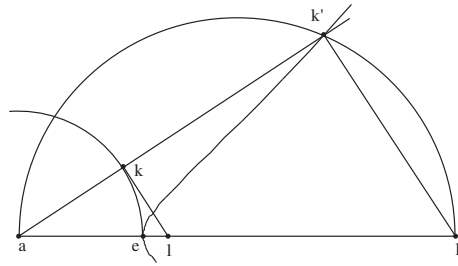


Figure 11.2. Solving the  $2\mu$  mean proportionals problems by relying on a curve traced by a proportions compass.

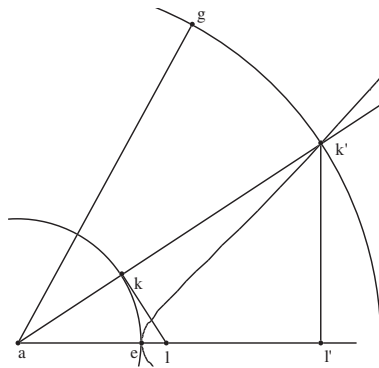


Figure 11.3. Solving the  $2\mu - 1$  mean proportionals problems by relying on a curve traced by a proportions compass.

compass with  $AK_0 = a$ , in such a way that  $e$  coincides with its origin. With center  $a$ , trace a circle of radius  $ag$  cutting this curve at  $k'$ . Join  $a$  to  $k'$  and, with center  $a$ , trace the circle of radius  $ae$  cutting  $ak'$  at  $k$ . Produce  $ae$  and trace  $lk$  and  $l'k'$ , respectively perpendicular to  $ak'$  and  $al'$ . Then  $al$  and  $al'$  are respectively equal to the smallest and the greatest of the  $2\mu - 1$  sought-after mean proportionals.  $2\mu - 3$  can be constructed as fourth proportionals.

One understands why, for Descartes, there is neither an “easier” way to solve the mean proportionals problems nor a “more evident” proof that their solution is sound [Descartes 1637, 370; Descartes [AT], VI, 442–443]. But this is not all, since, despite this, he famously adds that “it would be a mistake in geometry” to apply this solution for finding two, four, or six mean proportionals, since, “for the construction of any problem [. . .] we should choose with care the simplest [curve],” and these mean proportionals can be found through curves of a “simpler genus” than those traced by proportions compasses [Descartes 1637, 369–371; Descartes [AT], VI, 442–444; Bos 2001, 357–359].<sup>48</sup> Descartes is generalizing Pappus’s simplicity precept here [Pappus [CMH], I, 270–273; Bos 2001, 48–50], and—by assuming that any curve to be admitted in geometry is expressed by a two-variable polynomial equation—he measures the simplicity of these curves through the degree of their equations. Let us see how his precept applies to the present case.

<sup>48</sup> The translations of quotations from the *Géométrie* are mine, but are based either on Smith and Latham’s translation [Descartes [GDSL]], or on Bos’s or others’ translations offered in the secondary sources I refer to.

If referred to orthogonal lineal coordinates whose origin and axis coincide respectively with the pole A and the ruler AP of a proportions compass with  $AK_0 = AE = a$ , the curves traced by the points  $K_i$  ( $i = 1, 2, \dots$ ) of this compass have equation  $x^{4i} = a^2(x^2 + y^2)^{2i-1}$ . The solution of the two mean proportionals problem provided by the former of the two previous constructions relies then on a quadric, whereas Menaechmus's solution (references are given in Section 2.3.2) relies on two conics.

Something similar happens for the four and six mean proportionals problems. At the end of the *Géométrie* [Descartes 1637, 411–412; Descartes [AT], VI, 483–484; Bos 2001, 368–372], Descartes shows how to solve the former by relying on a circle and a cubic—the so called Cartesian parabola, introduced in the second book [Descartes 1637, 309, 322, 337, 343; Descartes [AT], VI, 381–382, 395, 408–409, 415]—whereas the solution provided by the first of the two previous constructions relies on a curve of degree 8. He does not show how to solve the latter problem by relying on curves simpler than the curve of degree 12 involved in the first of the two previous constructions, but he seems to think that his solution of the former problem can be generalized.

This is quite a difficult matter. Fortunately, we do not need to enter into it, since two things are immediately clear: (i) for any positive integer  $\mu$ , the  $2\mu$  mean proportionals problem can be solved by relying on two curves of equations  $yx^\mu = a^\mu$  and  $a^\mu by = x^{\mu+1}$ , respectively; (ii) if  $h$ ,  $p$ , and  $q$  are three positive integers such that  $h = pq$ , the  $h - 1$  mean proportionals problem can be reduced to the  $p - 1$  and  $q - 1$  mean proportionals ones.<sup>49</sup>

From (ii), it follows that, for any positive integer  $\mu$ , the  $2\mu - 1$  mean proportionals problem can be reduced to the single mean proportional and the  $\mu - 1$  mean proportionals ones, and then, by reiteration, either to the single mean proportional problem alone, or to the single mean proportional and the  $2v$  mean proportionals problems, for some positive integer  $v$  such that  $2v < 2\mu - 1$ . Jointly with (i), this entails that, whatever the positive integer  $n$  might be, solving the  $n$  mean proportionals problem by relying on a curve traced by a proportions compass does not comply with Descartes's simplicity precept.<sup>50</sup>

<sup>49</sup> This is easy to explain. Let  $a$  and  $b$  be the two given segments and suppose  $a < b$ . If  $x$  is the smallest of  $p - 1$  mean proportionals between them, the smallest of  $q - 1$  mean proportionals between  $a$  and  $x$ , is also the smallest of  $h - 1$  mean proportionals between  $a$  and  $b$ . The reason is obvious: if the  $p - 1$  mean proportionals between  $a$  and  $b$  are  $x, y, \dots, w$  ( $x < y < \dots < w$ ), by introducing  $q - 1$  mean proportionals between  $a$  and  $x$ , other  $q - 1$  ones between  $x$  and  $y, \dots$ , and finally other  $q - 1$  ones between  $w$  and  $b$ , one gets  $(q - 1)p + p - 1 = pq - 1 = h - 1$  mean proportionals between  $a$  and  $b$ .

<sup>50</sup> The reason is evident. Suppose that  $n$  is even, that is,  $n = 2\mu$ , for some positive integer  $\mu$ . Then, the curve traced by a proportions compass that enters into the solution of the  $n$  mean proportionals problem has equation  $x^{4\mu} = a^2(x^2 + y^2)^{2\mu-1}$  and is then a curve of degree  $4\mu = 2n$ . On the other hand, from (i) it follows that this same problem can also be solved by appealing to two curves of degree  $\mu + 1 = (n + 2)/2$ . Better, if  $n + 1$  is not prime, from (ii) it also follows that this problem can be reduced to the  $p - 1$  and  $q - 1$  mean proportionals ones, where  $p$  and  $q$  are such that  $pq = n + 1$ . Suppose, instead that  $n$  is odd, that is,  $n = 2\mu - 1$ , for some positive integer  $\mu$ . The curve traced by a proportions compass that enters into the solution of the  $n$  mean proportionals problem has equation  $x^{4\mu} = a^2(x^2 + y^2)^{2\mu-1}$  again, and is then a curve of degree  $4\mu = 2(n + 1)$ . On the other hand, from (ii) it follows that this same problem can be reduced to the single mean proportional and the  $\mu - 1 = (n - 1)/2$  ones. If this last number is odd, the reduction can continue in the same way. If it is even, then from (i) it follows that the problems can be solved by appealing to two curves of degree  $(n + 3)/4$ .

Two distinct criteria are then opposed to each other concerning the choice of the appropriate solution for the mean proportionals problems: one of easiness, another of simplicity. The former prescribes solving these problems by relying on curves traced by proportions compasses, which are easy to conceive and use; the latter prescribes relying on curves of the lower possible degree, which (a few particular cases aside) are quite difficult to determine. According to Descartes, to choose the former solution is a “mistake.” But then, why does he mention the proportions compass twice in the *Géométrie*? The answer is that this mistake is one “in geometry”: by choosing the former solution, one makes a mistake, but still appeals to constructions and curves that should be admitted in geometry. Hence, the easiness of such a solution can be exploited both to illustrate the exactness norms relative to such curves,<sup>51</sup> and to show that meeting these norms does not assure simplicity. The former point is made in Book II, the latter in Book III. For my present purpose, only the former is relevant.<sup>52</sup>

### 3.2. Ruler, compass, and reiteration

More precisely, Descartes makes this point immediately after the discussion of Pappus’s classification of geometrical problems into “plane,” “solid,” and “linelike” ones, according to whether their solution requires only straight lines (or segments) and circles, also requires conics, or needs, as Descartes says, “more composed lines,” or, in Pappus’s parlance, “lines having a varied and more convoluted origin” [Descartes 1637, 315–317; Descartes [AT], VI, 388–390; Pappus [CMH], I, 270–271; Bos 2001, 37–48]. During this discussion Descartes remarks that among line-like problems there are some that can be solved by relying on curves that share an essential feature with straight lines, circles and conics. Hence, the appropriate classification is not Pappus’s, but a more complex one concerned with curves, rather than with problems: one should first distinguish those curves that share this feature from those that do not, and then classify the former. Whereas geometricity requires using only the former, simplicity requires using, among them, those of the lower genus. Descartes also criticizes the “ancients” for having termed ‘mechanical’ any curves other than circles and conics. He argues that this denomination cannot be justified by advancing that “some sort of mechanical instruments [*machines*] has to be used to describe them,” since circles and straight lines also “cannot be described on paper without the use of a compass and a ruler, which may also be termed ‘mechanical instruments’” [Descartes 1637, 315; Descartes [AT], VI, 388].

Insofar as Descartes seems to take for granted that rulers and compasses have to be used in accordance with the clauses of elementary constructions, his point seems to be that obtaining circles and straight lines (or better, segments) requires elementary constructions. This is obviously not the same as arguing that elementary constructions are enough for constructing all circles and straight lines (or segments) that have to be constructed in order to solve a geometrical problem. Still, Descartes seems to imply that the very last step in the construction of circles and straight lines (or segments) has to depend on the application of

<sup>51</sup> For Serfati [1993, 219–220], the curves traced by proportions compasses are “exemplars” of those that Descartes admits in geometry.

<sup>52</sup> Among the large literature concerning the latter point, let me point out the recent contribution of Lützen [2010].

one of the constructive clauses of elementary constructions.<sup>53</sup> Hence, it is only by admitting the possibility of obtaining some non-EPG objects, namely some curves other than circles, that one can go, for him, beyond the limits of these constructions. This is the same as denying the rightfulness of quasi-elementary constructions. Descartes seems thus to consider that these constructions have either to be recast under the form of strictly nonelementary ones, or *ipso facto* discarded.<sup>54</sup>

His insistence on instruments used in the tracing way is not enough, however, to discard strictly nonelementary constructions that do not appeal to instruments, since the curves involved in these constructions could be also traceable through appropriate instruments. Hence, according to Descartes, the search for new exactness norms to be added to those of EPG results in a double purpose: to identify an appropriate class of instruments to be used in the tracing way for obtaining curves other than circles; to establish whether some curves obtained in some other ways could also be traced by these instruments. This double purpose results, in turn, from a double reduction: the question of fixing the nonelementary constructions to be admitted in geometry is first reduced to the question of identifying the curves other than circles that are to be admitted in geometry; this question is then reduced to that of identifying a class of instruments that, when used in the tracing way, trace curves that are admitted in geometry just because they can be so traced. Descartes famously terms these curves ‘geometrical’ [Descartes 1637, 319; Descartes [AT], VI, 392].<sup>55</sup> For short, let us also call ‘geometrical linkages’ the instruments to be used to trace these same curves (arguments justifying this denomination will be offered later).

<sup>53</sup> An example can be useful to better explain my understanding of Descartes’s claim. Compare Viète’s and Pappus solution of the angle trisection problem (both related in Section 2.3.1). They both involve a nonelementary construction. But the reasons the constructions involved in these solutions are nonelementary are essentially different from each other. The construction involved in the former is nonelementary because it includes the construction of a segment (the segment FE: Fig. 1) under the supposition that some EPG objects are given (the circle of center B and radius BE, the point E on it, and the segment BD), these objects being such that no constructive clause of elementary constructions licenses constructing this segment under the supposition that they are given. The construction involved in the latter is nonelementary because it includes the construction of an hyperbola, which is not an EPG object (cf. footnote 39, above). Once this hyperbola is constructed, its intersection point D with the relevant circle (Fig. 3) is *ipso facto* constructed, as occurs in elementary constructions for the intersection points of segments and circles. And, once this point is given, the construction continues as an elementary one. Namely, the segments AD and EF, on which the trisection depends, are constructed according to the constructive clauses of elementary constructions. Hence, whereas Viète’s construction includes a step in which a segment is constructed according to a constructive clause which is not included among those of elementary constructions, the very last step in the construction of any segment and circle involved in Pappus’s construction depends on the application of one of the constructive clauses of elementary constructions. According to my understanding, Descartes’s point is that constructions such as the former are not admissible.

<sup>54</sup> I shall come back to the possible reasons for this exclusion in Section 3.3.

<sup>55</sup> This second reduction produces an asymmetry first remarked by Mancosu [Mancosu 2007, 114–121, esp. 117; Mancosu 1996, 71–79] and discussed in Mancosu and Arana [2010]: to show how to trace a curve through a geometrical linkage is enough to establish that it is geometrical, but ignorance regarding the possibility of tracing a curve through such a linkage is not enough to establish that it is not so.



Here is how Descartes characterizes these instruments [Descartes 1637, 316–317; Descartes [AT], VI, 389–390]<sup>56</sup>:

[...] mais il est, ce me semble, tres clair que, prenant, comme on fait, pour Geometrique ce qui est precis et exact, et pour Mechanique ce qui ne l'est pas; et considérant la Geometrie comme une science qui enseigne generalement a connoistre les mesures de tous les cors; on n'en doit pas plutost exclure les lignes les plus composées que les plus simples, pourvû qu'on les puisse imaginer estre descrites par un mouvement continu, ou par plusieurs qui s'entresuivent et dont les derniers soient entierement réglés par ceux qui les precedent: car, par ce moyen, on peut toujours avoir une connaissance exacte de leur mesure. [...] la Spirale, la Quadratrice, et semblables [...] n'appartiennent veritablement qu'aux Mecaniques et ne sont point du nombre de celles que je pense devoir icy estre receues, a cause qu'on les imagine descrites par deux mouvements separés et qui n'ont entre eux aucun rapport qu'on puisse mesurer exactement [...].

Descartes's discussion of proportions compasses in Book II is intended to illustrate this passage. Hence, it is not only natural to wonder how this characterization should be understood, but also why this compass illustrates it.<sup>57</sup>

All the responses offered to these questions agree on a fundamental statement that I also share: Descartes requires that geometrical linkages be such that the motions of all their parts depend on a unique principal motion which determines any other motion, including those of the tracing points.<sup>58</sup>

This condition is clearly illustrated by proportions compasses (whose principal motion is the rotation of ruler AQ). But these compasses also meet other conditions. One of them is

<sup>56</sup> Because of the relevance of this passage, I prefer to quote it in the original French; slightly different translations are offered in Descartes [[GDSL], 43–44; Bos 2001, 338 and 341; Mancosu 1996, 71–72].

<sup>57</sup> The interplay between general standards of rational conceivability and intrinsically geometrical requirements (including faithfulness to the spirit of EPG) that is typical to Descartes's geometry is evident in his characterization of geometrical linkages and curves. Characterizing geometrical linkages is certainly not enough to explain why Descartes considers curves traced by them to be admissible in an "exact" science such as (pure) geometry, that is, why he considers them to be exactly conceivable: this is a question about Descartes's epistemology that cannot be settled by considering only his geometry. But, on the other hand, no account of his general notion of rational conceivability can be enough to understand his characterization of geometrical linkages and curves. Two examples are enough to explain why. Domski [2009, 123] emphasizes the role played in Descartes's geometry by "a standard of intelligibility grounded on simple and clearly conceivable motions," and suggests that geometrical curves are those that can be obtained through an "intelligible motion." Arana [forthcoming] suggests, instead, that for Descartes "constructed objects are known best when the construction is carried out in a way that is fully present to the attentive constructing mind." Both suggestions have value. But they leave entirely open the problem of understanding what makes, for Descartes, some motions simple and clearly conceivable in geometry, or some geometrical constructions fully present to the attentive mind. My upcoming discussion could be taken as a tentative clarification of these matters.

<sup>58</sup> A similar point is made in a famous letter to Beeckman of 16 March 1619, where Descartes claims that the curves described by his instruments "result from one single motion," whereas other curves such as the quadratrix are "generated by different motions not subordinate to one other" [Descartes [AT], X, 157; Bos 2001, 231]. Using the terminology of modern mathematical analysis, one could say that in the former case, the relevant single motion is the independent motion among those that the different parts of the instrument are submitted to, in the same sense in which  $x$  is the independent variable relative to a function  $y = f(x)$ .

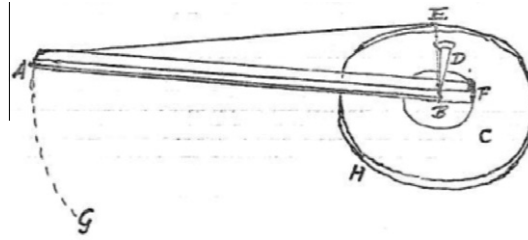


Figure 12. Huygens's instrument for tracing spirals.

that, if they move, each of their motions can only follow a unique and perfectly determinate trajectory, to the effect that the speed and direction of these motions have no influence on the curves they trace (since these curves are just the trajectories that the motions of some points taken on the compasses are constrained to follow, if these compasses move).<sup>59</sup>

Descartes does not mention this condition—probably because he (wrongly) takes it to be entailed by the previous one—but he clearly requires that all geometrical linkages meet it. It follows that the curves traced by each of the tracing points of a geometrical linkage are, by definition, univocally determined regardless of the way this same linkage is set in motion. This is also the case for straight lines and circles, if they are taken to be traced by rulers and compasses (respectively conceived as fixed bars on which a tracing point moves, and as rotating bars on which a tracing point is fixed). I suggest that this is the essential feature that, according to Descartes, geometrical curves share with straight lines and circles: they have to be traceable by instruments so conceived that, if they move, their tracing points are constrained to follow some determinate trajectories that are independent of their actual motion (that is, both of the fact that they actually move, and of the direction and speed with which they possibly do), to the effect that this is also the case of the curves they trace, which are nothing but these same trajectories.<sup>60</sup> Geometrical linkages have then to be instruments such as these, i.e., mere tools capable of fixing such trajectories.

But, though necessary, this and the previous condition stated in Descartes's quote are still not sufficient to characterize geometrical linkages. To understand why, consider the instrument for tracing spirals that Huygens sketched in his notebook in 1650, possibly after having heard about it from Descartes himself [Bos 2001, 345, 347–349; Mancousu and Arana 2010, Sect. 3; Huygens's sketch is reproduced in Fig. 12, which is taken from Bos 2001, 348]. A ruler AF is left free to rotate around a fixed pole B, in which a fixed disk C is centered; a string is attached to the disk at the top extremity E of it, goes up to the moving extremity A of the ruler, and then comes back along the ruler itself up to a tracing pin D that is left free to slide on this ruler. Initially, the ruler stands horizontally on the left of the disk and the tracing pin is placed in the pole. While it rotates counterclockwise, the string winds up around the disk and pulls the tracing pin, so that it traces an arc of spiral.

This instrument meets the condition for geometrical instruments explicitly stated in Descartes's passage quoted above. Its principal motion is the rotation of the ruler AF, and it is then constrained to follow a determinate trajectory, indicated for example by the circle

<sup>59</sup> Proportions compasses seem, however, to have an initial position from which the rotating ruler can only move in one direction. I shall come back to this matter later.

<sup>60</sup> Serfati [1993, 227–228] has similarly argued that in the “generation” of curves through geometrical linkages only “the automatism” of such an instrument is at issue, whereas the generation of spirals and quadratrices require “a thinking subject that, at any instant, brings together two movements [...] in her/his hand.”

described by the point A during this rotation. Moreover, the speed of this motion has no influence on the trajectory followed by the tracing pin D, which is the curve traced by this instrument. Still, this trajectory, and thus this curve, are not independent of the direction with which the principal motion describes its trajectory, since, for the pin D to move, the ruler AF has to turn counterclockwise. But suppose that the string is replaced by a wire so conceived that the tracing pin D also moves if the ruler AF rotates clockwise by going either forward or backward along this ruler according whether this same ruler rotates counterclockwise or clockwise. The new instrument that is so gotten is such that the trajectory followed by the tracing pin D, and then the curve it traces, are independent of its actual motion.

Still, Descartes would have not considered it as a geometrical linkage: it is likely that he knew Huygens's original instrument and nevertheless considered the spiral not to be a geometrical curve; and it is also likely that he would have not changed his mind if he had imagined the previous modification.

Though Descartes never mentions Huygens's instrument, he considers instruments involving strings, and he argues that the curves they trace should be taken to be geometrical if strings are used "to determine the equality or difference of two or more straight lines which can be drawn from each point of the sought-after curve to certain other points or toward other lines at certain angles." He also adds that "one cannot accept [in geometry] any lines which are like strings, that is to say which become sometimes straight and sometimes curved" [Descartes 1637, 340; Descartes [AT], VI, 412; Bos 2001, 347; Mancosu 2007, 118].

The string involved in Huygens's instrument is not used as required in the former of these passages, and rather behaves as is said in the latter. Many scholars have then argued that Descartes does not consider this or similar instruments to be geometrical linkages just for this reason. For Descartes, curves that "become sometimes straight and sometimes curved" are not geometrical because "the proportion between straight lines and curves [...] is not known, and [...] will never be so to man" [Descartes 1637, 340; Descartes [AT], VI, 412; Bos 2001, 347; Mancosu 2007, 118]. Hence, these scholars argue that Descartes's motivation for discarding these instruments depends on his agreement with this old Aristotelian dogma: he would admit that segments and arcs of curves are incommensurable magnitudes or, at least, magnitudes that stay to each other in an exactly unknowable proportion, and he would then discard these instruments, alleging that the proportion between the straight and curved parts of their strings is exactly unknowable. This is Bos's view. Moreover, Bos thinks that Descartes's "separation between geometrical and non-geometrical curves [...] rested ultimately on his conviction that proportions between curves and straight lengths cannot be known exactly" [Bos 2001, 342, 349].

This view is problematic. Mancosu [2004, 119; cf. also Mancosu 1996, 77] has challenged it by observing that "the algebraic rectification of certain algebraic curves in the 1650s did not undermine the foundations of Descartes's *Geometry*." More specifically, one could also remark that there is no need for the proportion of the straight and curved parts of the string entering into Huygens's instrument to be known in order for this instrument to work. It is only necessary to know this proportion in order to characterize the curve that is traced independently of the instrument itself. But to require that geometrical linkages trace curves that could be characterized independently of them would be the same as inverting the definitional order between geometrical linkages and curves, characterizing the former on the basis of the latter, rather than vice versa. Hence, either Descartes's characterization of geometrical curves does not rest, in fact, on his characterization of geometrical linkages, or the previous reason for discarding instruments such as Huygens's is not sound.

I take the latter to be true, and suggest that the reason that Descartes discards these instruments is another one. To understand it, consider proportions compasses again. Conceived as material devices, they cannot but be composed of a finite number of finite rulers. Still, they can also be conceived as abstract systems, i.e., as appropriate configurations of an infinite number of straight lines that move by meeting some incidence conditions and without any force being exerted. And, if they are so conceived, AQ can move indefinitely on both sides of AP so that points  $K_i$  ( $i = 1, 2, \dots$ ) trace infinitely many curves infinitely extended. The previous discussion of Huygens's instrument and of its modified version should make clear, instead, that they cannot be so conceived, because their mechanical nature is essential for them to work. Not only does their working depend on the physical properties of their strings or wires, and not only must their ruler AF have an extremity or some sort of hub around which such a string or wire passes, but they are also so designed to work only insofar as some forces are exerted by and upon their components. Moreover, there is no room to suppose that their motion is indefinitely continued so as to trace an entire spiral.

There is thus an essential difference between instruments such as Huygens's and proportions compasses: the latter can be conceived as purely geometrical systems that are taken to move and to trace entire curves because of their motion; the former are intrinsically mechanical devices that cannot but trace finite arcs of curves. I suggest that, for Descartes, geometrical linkages are instruments like the latter and not like the former: they are—or at least they can be conceived as—moving configurations of geometrical objects.

It is enough to admit that the term 'line' refers to geometrical objects (which is quite natural) to recognize this idea rather explicitly expressed in something Descartes writes before mentioning the proportions compasses for the first time [Descartes 1637, 316; Descartes [AT], VI, 389; Bos 2001, 338]: "nothing else needs to be supposed to trace all the curves that I purport to introduce here than that two or several lines can be moved one by the other and that their intersection mark some other ones [...]." In the diagrams included in the *Géométrie*, rulers entering into geometrical linkages are represented by double strokes, which evokes their thickness. Still, this passage seems to say that they are—or at least can be conceived as—nothing but straight lines or segments.

Another piece of textual evidence supporting my suggestion is offered by what Descartes writes a little later, concerning the way the ancients used the term 'mechanical curves' [Descartes 1637, 315–316; Descartes [AT], VI, 389]. He argues that this use cannot be justified by observing that the curves the ancients termed 'mechanical' are traced by instruments that are too "complicated [*composé*]" to be "right [*iustes*]." He then adds that this should rather suggest rejecting these curves from mechanics, since it is there that "the rightness of the works made with hands is desired," whereas in geometry "only the rightness of reasoning is pursued," and rightness of reasoning can be as "perfect" about the curves traced by such instruments as it is about straight lines, circles, and conics. This suggests that, for Descartes, the relevant properties of the instruments tracing geometrical curves and of these same curves do not depend on the material features of these instruments, but rather on the way they are conceived by "reasoning."

If this double textual evidence is taken to be weak, consider that it is a matter of fact that Descartes rules out instruments including strings working as in Huygens's and that the reason usually invoked to justify this exclusion is both implausible and unsound, whereas the reason I suggest does not seem to be so.

This reason agrees, moreover, with Descartes's admission that some instruments, including strings, can trace geometrical curves. Descartes's point is, indeed, just this, and not that

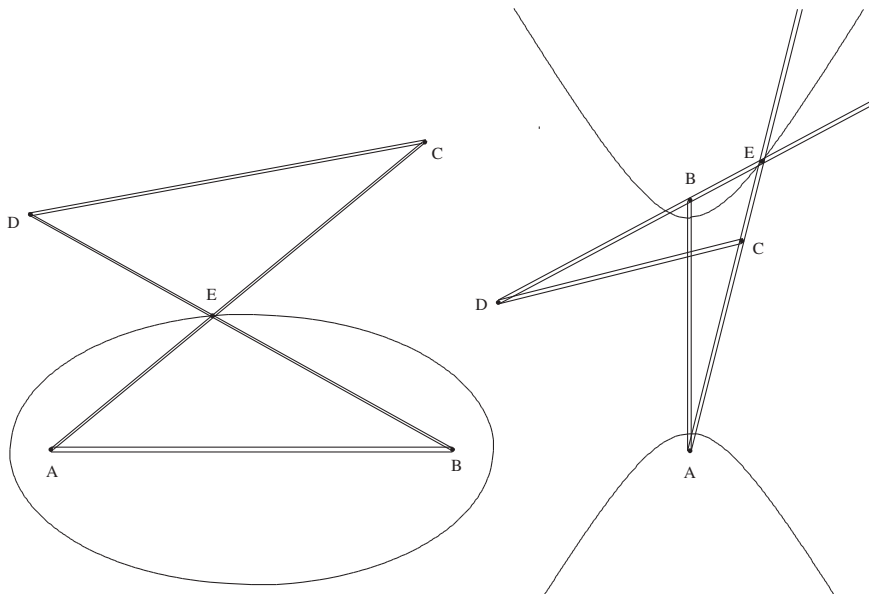


Figure 13. Instruments for tracing ellipses and hyperbolae.

these instruments are geometrical linkages. As examples, he mentions the instruments evoked in the *Dioptrique* “in order to explain the ellipse and the hyperbola” [Descartes 1637, 340; Descartes [AT], VI, 412]. These are the instruments entering into the gardener’s constructions of these curves. Descartes terms them “very coarse and not very exact,” but still maintains that they are such as “to make [...] the nature [of these curves] better known” [Descartes 1637, 89–90, 100–101; Descartes [AT], VI, 166, 176]. This suggests that Descartes takes these constructions to be capable of fixing the nature of these curves, but still not to be admissible in geometry, as such. I explain this as follows: these constructions rely on intrinsically mechanical devices but suggest two instruments involving no string, which also trace these curves and meet all the previous conditions for being geometrical linkages. These are two anti-parallelograms ABEC (Fig. 13) whose side AB is fixed, while sides AC and BD rotate together around A and B, being linked by side DC, so that these curves are traced by their intersection point E. Descartes’s claim seems thus to be that some curves traced by instruments including strings are geometrical not because these instruments are themselves geometrical linkages, but because they suggest geometrical linkages tracing these curves.<sup>61</sup>

To admit that geometrical linkages are—or can be conceived as—moving configurations of geometrical objects is of course not enough to characterize them. And, I argue that it is neither enough to add, according to what has been said above, that (a) the motions of all their parts depend on a unique principal motion; (b) they include one or more tracing points, and, if they move, these tracing points are constrained to follow some determinate trajectories which are independent of their actual motion, to the effect that this is also the case of the curves they trace, which are nothing but these same trajectories. What makes a moving configuration meeting these conditions a geometrical linkage (i.e., an instrument that traces a geometrical curve in Descartes’s sense) is, I hold, that it can be obtained

<sup>61</sup> A similar point is made by Molland [1976, 42].

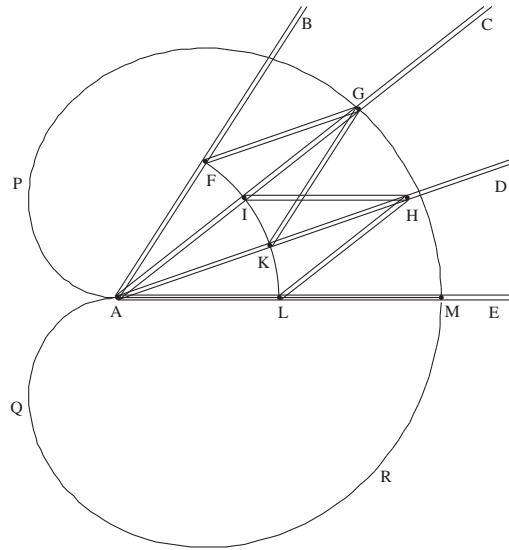


Figure 14. Trisecting compass.

through a licensed construction.<sup>62</sup> More precisely, I argue that such a moving configuration is a geometrical linkage if and only if it can be constructed in an arbitrary position through a licensed construction. This condition needs clarification.

A first point to be clarified is concerned with the requirement that such a moving configuration be constructed in an arbitrary position. This means that what has to be constructed is the fixed configuration of geometrical objects that constitutes the position it takes when an arbitrary respective position of the objects directly involved in its principal motion is chosen, in such a way that these objects do not coincide with each other, and its invariant components are allowed to meet the conditions they are possibly required to meet. The construction has thus to begin from these objects taken in such an arbitrary position.

Consider two examples.

The first is provided by the proportions compass. The objects directly involved in its principal motion are the rulers AP and AQ (Fig. 11.1). Whatever their respective positions might be, the invariant distance  $AK_0$  can be determined as required. Hence, the construction of such a compass can start by choosing any arbitrary respective position of these rulers, provided they do not coincide.

The second example is less simple. It concerns the “compass” Descartes describes in the *Cogitationes Privatæ* (together with the proportions one) by suggesting using it to solve the angle trisection problem [Descartes [AT], X, 213–256, esp. 234–240; Serfati 1993, 205–212; Bos 2001, 237–245].<sup>63</sup> Call it ‘trisection compass.’ It includes four rulers AB, AC, AD, AE (Fig. 14) the first three of which are rotating around a common pole A, and the fourth

<sup>62</sup> It is quite usual to argue that Descartes’s linkages are “idealized instruments” to be “imagined” [cf. Molland 1976, 42]. My point is quite different. I hold them to be geometrical objects requiring (diagrammatic) construction, rather than mere imagination.

<sup>63</sup> The *Cogitationes Privatæ* date back to 1619. The same compass is also mentioned in the letter to Beeckman of March 26th of the same year [Descartes [AT], X, 154–160, esp. 154–156] as being part of a family of compasses, each of which is to be used to solve the  $n$ -section problem for a certain positive integer  $n$ .



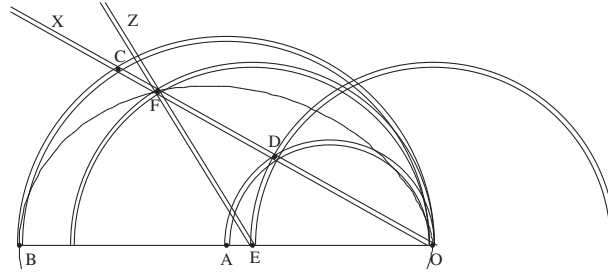


Figure 15. Villalpando's compass modified.

of which is fixed. On them, the points F, I, K, L are taken at equal distance from A, and at these points four other rulers, all equal to AL, are respectively attached so that those attached to F and K are also attached to the same point G sliding on AC, those attached to I and L are also attached to the same point H sliding on AD. If AL is such that AC is able to come full circle around A starting from being coincident with AE, while AC so turns, G traces the curve MGPQR.<sup>64</sup> This compass is certainly a geometrical linkage. Still, if one wanted to construct it starting by choosing an arbitrary respective position of rulers AE and AB, its construction would require either already having solved the angle trisection problem, or using the rulers composing this compass as physical devices that exert appropriate forces upon each other. If one starts, instead, by choosing an arbitrary respective position of rulers AE and AD and fixing L on the former so that AL is long enough, the compass can be constructed through an easy elementary construction. It is thus constructible in a licensed way only if its principal motion is taken to be the rotation of AD.

A second point to be clarified concerns the curves that the construction of geometrical linkages relies on, possibly. Return for this purpose to Villalpando's compass (Fig. 9.2), which is also, certainly, a geometrical linkage. But suppose that, instead of tracing Villalpando's locus through it, one wanted to trace this locus through another instrument so conceived that the tracing point F (Fig. 15) is at the intersection of two rotating rulers OX and EZ. This instrument should be such that OD = OE = EF. To ensure this, one could imagine equipping the instrument with two equal circles, respectively centered in E and O, and requiring that points D and F slide on them. But, as OD varies, the radii of these circles should also vary. If the instrument were conceived as a material device, it would thus be very hard to build, and it should in any case be made of some appropriate deformable stuff. If it were conceived as a moving configuration of geometrical objects all of whose motions depend on a principal one, it should include some system of lines used to transform the rotation of OX or EZ into the increasing of the radii of these circles. Villalpando's compass (Fig. 9.2) works in a much simpler way, since it transmits its principal motion (the rotation of IM or JM) into the motion of F without relying on any circle. This shows that there is no need to appeal to changing circles in order to assure that, while a linkage moves, some segments included in it remain equal, though varying in length. One can rather use appropriate

<sup>64</sup> This is a sextic curve with equation  $4a^4x^2 = (x^2 + y^2)(x^2 + y^2 - 2a^2)^2$ , with respect to orthogonal coordinates of origin A and axis AE, with  $a = AL$  [Bos 2001, 238, footnote 20]. The trisecting compass stands to the angle trisection problem in the same relation as the proportions compass stands to the mean proportionals problems: it can be used to solve the former problem both in the pointing and in the tracing way, but this problem can also be solved (as said in Section 2.3.1) by relying on simpler curves, namely on circles and conics.



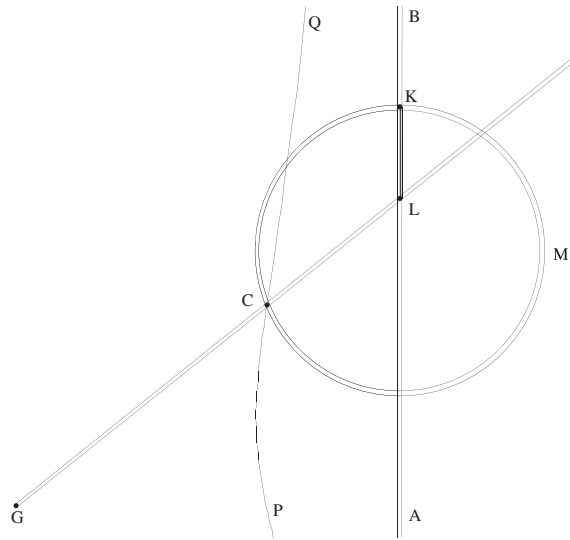


Figure 16.1. Turning ruler with a moving circle.

configurations of rotating segments or straight lines meeting some conditions of orthogonality.

But there are other purposes for which circles can enter into geometrical linkages. An example is gotten if Villalpando's compass is transformed by replacing the ruler IM (Fig. 9.2) with a circle of diameter IJ on which M is required to slide. This transformation has no influence on the curve which is traced. The new instrument is thus a geometrical linkage including a fixed circle. Hence, the inverse transformation shows that fixed circles can enter into geometrical linkages in such a way that they can be replaced by rulers without influence on the curve which is traced. But there are also cases of geometrical linkages including circles that cannot be replaced by other components without influence on the curve which is traced. An example including a circle moving rigidly is provided by Descartes himself [Descartes 1637, 322; Descartes [AT], VI, 395]: a ruler GL (Fig. 16.1) rotates around a fixed pole G while the point L slides on it in such a way as to stay at its intersection with another fixed ruler AB; to L is attached another ruler LK that slides with this point on AB; to K is attached a circle KCM; the intersection point C of this circle and the ruler GL traces a curve PCQ which is a conchoid of a straight line.

Descartes comes to this example by modifying a simpler instrument including, instead of the circle KCM, a straight line KC (Fig. 16.2) forming a fixed acute angle with LK, and claims that an infinity of similar linkages can be reached from this basic one by replacing this straight line with any geometrical curve [Descartes 1637, 319–323; Descartes [AT], VI, 392–395; Serfati 1993, 220–221; Serfati 2002; Bos 2001, 278–281]. Following Bos, call these linkages 'turning rulers with a moving curve.' The basic one traces a hyperbola; Descartes first suggests replacing the straight line KC with this same hyperbola, then continues by suggesting replacing it either with a circle (as said) or with a parabola, and finally observes that if the linkage includes a geometrical curve of the  $i$ th genus ( $i = 1, 2, \dots$ ), it traces a geometrical curve of the  $(i + 1)$ th genus.

This is wrong [Serfati 2002], but shows that Descartes admits that geometrical linkages can include circles or any other sort of geometrical curves moving rigidly. More generally, I advance that geometrical linkages can include circles or any other sort of geometrical curves if and only if these are fixed or move rigidly.

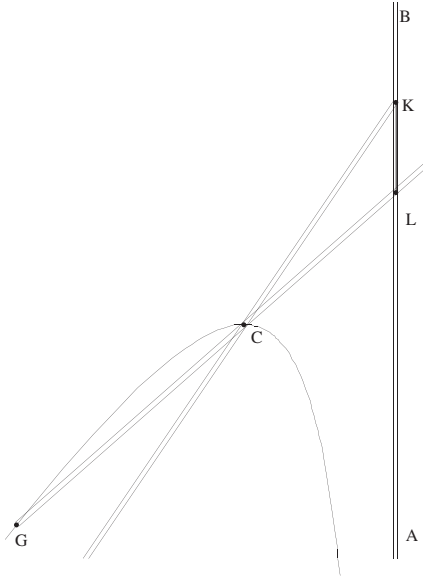


Figure 16.2. Turning ruler with a moving straight line.

All this finally suggests a necessary and sufficient characterization of geometrical linkages. They are moving configurations of geometrical objects that meet the conditions (a) and (b) stated above, possibly include, besides straight lines (or segments), also circles and geometrical curves which are fixed or move rigidly, and can be obtained, together with the geometrical curves they trace, according to the following recursive procedure.

Term ‘elementary’ the geometrical linkages that can be constructed in an arbitrary position through an elementary construction. Say that the curves traced by them are obtained by a construction of type  $\mathcal{C}^{[0]}$ . Examples of elementary linkages are usual compasses, as well as conchoid, proportions, trisection, and Villalpando’s compasses, and turning rulers with a moving curve where the moving curve is either a straight line or a circle.

The curves obtained by a construction of type  $\mathcal{C}^{[0]}$  can enter into new nonelementary geometrical linkages. These can be constructed in an arbitrary position through a nonelementary construction including the construction of these curves. This last construction requires that appropriate elementary geometrical linkages be constructed starting from appropriate given objects in such a position that they trace these curves in an appropriate position. Say that the curves traced by such new geometrical linkages are obtained by a construction of type  $\mathcal{C}^{[1]}$ . An example is provided by the already mentioned Cartesian parabola. This is traced by a turning ruler with a moving curve where the moving curve is a usual parabola of axis AB. The construction of this last parabola requires that an elementary linkage tracing it be constructed so as to trace this curve in such a position that its axis coincides with AB and its vertex with K.

The procedure can continue indefinitely in the same way: the curves obtained by a construction of type  $\mathcal{C}^{[1]}$  can enter into further geometrical linkages tracing curves that are then obtained by a construction of type  $\mathcal{C}^{[2]}$ , etc.

If these clauses are combined with those of elementary construction—by admitting, for example, that a point can be constructed by intersection of a geometrical curve and a given

segment or circle—one gets a new sort of diagrammatic nonelementary construction.<sup>65</sup> I suggest terming constructions obtained in this way ‘constructions by ruler, compass, and reiteration.’ They allow one to obtain two essentially distinct sorts of objects which populate Descartes’s geometry: the former are fixed objects, i.e., EPG objects and geometrical curves; the latter are moving objects, i.e., geometrical linkages. The former can interact with each other both in theorem-proving and in problem-solving. The latter are not supposed to interact with each other, and a single geometrical linkage interacts with some fixed objects only if it includes them, is constructed starting from them, or traces one of them.

Both fixed objects and geometrical linkages can be unconditional or conditional. Descartes is not explicit about the conditions under which they are available within his geometry. Still, it seems natural to admit that: (i) unconditional geometrical linkages and curves are available within Descartes’s geometry by definition (since they are geometrical linkages and curves just insofar as they can be constructed by ruler, compass and reiteration, which is what makes them available within Descartes’s geometry); (ii) unconditional EPG objects, conditional geometrical linkages, and conditional fixed objects are available within Descartes’s geometry if they can be constructed by ruler, compass, and reiteration, possibly starting from the given objects that the conditions involved in the corresponding concepts are concerned with. These are, I hold, the exactness norms that Descartes’s geometry is based on.

### 3.3. *Descartes on nonelementary constructions*

We can now explain Descartes’s attitude with respect to the six sorts of nonelementary constructions distinguished in Section 2.3.4.

A first point is clear: a construction involving curves described as trajectories of motions is licensed for him if and only if it is a construction by ruler, compass, and reiteration, or can be recast under the form of such a construction.

Concerning quasi-elementary constructions, I have already said that he rejects them from the very beginning. But it still remains to understand why.

For those relying on explicit stipulations or tacit admissions working as constructive clauses, the more plausible reason seems that he considers them as merely unjustified, insofar as they are based on *ad hoc* assumptions. Hence, either they can be appropriately recast, or they have to be discarded.

For those relying on instruments used in the pointing way, the situation is more delicate. As a matter of fact, Descartes admits geometrical curves by force of their being traceable through geometrical linkages. But then—one could plausibly wonder—why does he not also admit constructions appealing to other, similar, instruments, so conceived as to be used in the pointing way? Of course, to substantiate this question, one should explain the nature of these instruments, or, at least, the sense in which they might be considered as similar to geometrical linkages. Presumably, one could take these instruments to be, like geometrical linkages, nothing but moving configurations of geometrical objects, or at least to be such that one could

<sup>65</sup> The diagrammatic nature of these constructions is made manifest by the example of the nonelementary linkage generating the Cartesian parabola just described. It is clear, indeed, that the combination of the motion of the elementary linkage that traces the ordinary parabola that enters into this nonelementary linkage with the motion of this last linkage does not make such a parabola move rigidly, as it is assumed to do within this same linkage. Hence the motion of this ordinary parabola is not merely produced by the combination of the motions of the two linkages and can only be conceived, in fact, as the motion of the diagram that represents it.

conceive them as moving configurations of geometrical objects. But then, a possible reason for Descartes to reject constructions appealing to such instruments could be the following: in order to use moving configurations of geometrical objects in the pointing way one has to assign to the diagrams representing these objects a role quite different from the role played by diagrams both in elementary constructions and in constructions by ruler, compass, and reiteration. The conditions under which moving configurations of geometrical objects used in the pointing way would allow one to obtain the objects required could indeed not be imposed on these diagrams, but should rather be acknowledged by inspecting them.

A simple example is provided by Eratosthenes's solution to the two mean proportionals problem considered in Section 2.3.2. As I have noted in this section, one could imagine replacing the plates involved in this solution with genuine geometrical rectangles. But, as I also noted, this would make no relevant difference, since it would require using the diagrams representing these rectangles in such a way that the relevant conditions that these rectangles are supposed to meet are acknowledged by inspecting these diagrams rather than imposed on them.

The point here is that, even if this acknowledgment were conceived as a purely ideal procedure, it would remain essentially different from any procedure entering elementary constructions or constructions by ruler, compass and reiteration. Hence, supposing that Descartes's reason for rejecting quasi-EPG constructions appealing to instruments used in the pointing way was the previous one, his rejection of these constructions would depend on his purpose of staying as close as possible to Euclid's setting, and then excluding constructions structurally too different from EPG ones.

Consider now Descartes's attitude with respect to strictly nonelementary constructions.

Those involving univocally determined conics are of course admitted by Descartes, since they can be easily recast under the form of constructions by ruler, compass, and reiteration.

Something similar holds for generic pointwise constructions. These involve loci of points which are generated by other points or straight lines. Consider such a locus and suppose that it is the locus of a point that can be constructed, in an arbitrary position, by ruler, compass and reiteration, and it is generated by another point, so constructed in turn. It is then likely that the construction of these two points suggests a way for constructing a geometrical linkage to be used to trace this locus. The case of Villalpando's locus and compass provides an example. Descartes seems to think of a situation like this when he alleges to have "furnished a way to describe" a curve, by "having explained the way of finding an infinite number of points though which" it passes, and adds that "this way of finding a curve by finding several of its points at random applies only to those curves which can also be described by a regular and continuous motion" [Descartes 1637, 339–340; Descartes [AT], VI, 411–412; Bos 2001, 344–345; Domski 2009, 125–129]. One could then conclude that generic pointwise constructions of a curve are admitted by Descartes if and only if the curve is also traceable by a geometrical linkage suggested by the very construction.

The situation is different for strictly nonelementary constructions involving curves obtained by interpolation: however the points on which the interpolation is based are constructed, their construction provides no suggestion for constructing a geometrical linkage to be used to trace these curves. Hence, these constructions have to be rejected (if they cannot be appropriately recast). On this matter also, Descartes is explicit enough. In the *Géométrie*, he remarks that in these constructions "one does not find indifferently all points of the sought-after curve," so that, "strictly speaking, one does not find a [generic] point of it, that is, not one of those that which are so peculiarly points of it that they cannot be found except by means of it" [Descartes 1637, 340; Descartes [AT], VI, 411; Bos 2001, 344]. Even

more clearly, in writing to Mersenne on November 13th 1629, he argues that “although one could find an infinity of points through which the helix or the quadratrix must pass, one cannot find geometrically any one of those points which are necessary for the desired effect,” so that these curves “cannot be traced completely except by the intersection of two movements” [Descartes [AT], I, 71; Mancosu and Arana 2010, footnote 10 and the relevant quotation].

#### 4. Concluding remarks: Descartes’s geometry and EPG

Whereas the exactness norms stated at the end of Section 3.2 fix the bounds of Descartes’s geometrical ontology, the attitudes just described fix the constraints imposed by this ontology on the most common sorts of constructions in classical geometry. This is the same as fixing the bounds that have to be respected in solving geometrical problems. But it is much less than fixing a general method for solving geometrical problems, prescribing the appropriate way to solve each problem: such a method has to conform to these bounds, but these bounds are not enough to set it. Here is where simplicity and algebra come into account, thanks to the assumption that geometrical curves are just those that can be expressed, with respect to an appropriate system of lineal coordinates, by two-variable polynomial equations.

This assumption also suggests a way to restate the bound of Descartes’s geometrical ontology in a more convenient way, which, as a matter of fact, has been historically prominent after Descartes. This leads to the problem of the relations between constructions by rulers, compass, and reiteration and Descartes’s algebra. This problem has been differently stated and tackled many times, and I cannot enter into it. My present enquiry can suggest, at most, a way to formulate it anew.

Typically, unconditional geometrical linkages are defined by Descartes by describing how to construct them. Their definition comes thus together with the proof that these linkages are available within this geometry. Hence, if unconditional geometrical curves of a certain sort are defined as the curves traced by a certain sort of unconditional geometrical linkages so defined, the definition of these curves also goes together with the proof that they are available within this geometry. The same happens if unconditional geometrical curves of a certain sort are defined as the curves expressed, with respect to any system of lineal coordinates, by equations of a certain form. In both cases, no need arises of stating and solving problems concerned with the concepts of the different sorts of unconditional geometrical curves.

For conditional geometrical curves, things go in a slightly different way. They can be defined either by specifying conditional geometrical linkages purporting to trace them, or by providing equations including coefficients which refer to some supposedly given segments, and purporting to express these curves with respect to appropriate systems of lineal coordinates. In the former case, to prove that these curves are available within Descartes’s geometry, one has merely to prove that the relevant linkages are so. In the latter, one has to prove that the relevant system of coordinates is also available within this geometry (which means that its origin, its axis, and its angle are so), and the relevant equations can be determined.

As an example, consider the geometrical curves traced by proportions compasses. If they are unconditional, they can be defined either as the curves traced by unconditional proportions compasses, or as curves expressed, with respect to any system of orthogonal lineal coordinates, by equations of the form  $x^{4n} = a^2(x^2 + y^2)^{2n-1}$ , where ‘ $n$ ’ stands for any natural number and ‘ $a$ ’ stands for any segment. It is enough to define these curves in one of these

two ways to warrant that they are available within Descartes's geometry. If they are conditional, they can be defined either as the curves traced by conditional proportions compasses—i.e., a proportions compasses whose element  $AK_0$  (Fig. 11.1) is required to be equal to a supposedly given segment and which is possibly so placed so as to trace this curve in an appropriate position—or as the curves expressed, with respect to a certain determinate system of orthogonal lineal coordinates, by equations of the form  $x^{4n} = a^2(x^2 + y^2)^{2n-1}$ , where 'n' stands for any natural number and 'a' denotes a supposedly given segment. To prove that these curves are available within Descartes's geometry, one has to prove either that these conditional proportions compasses are available within Descartes's geometry, or that this is the case for this system of coordinates and for the segment denoted by 'a.'

These considerations are enough to show that the relation between definitions and exactness norms of geometrical objects is different in Descartes's geometry than in EPG. Whereas problems complying with the ontological function<sup>66</sup> are indispensable ingredients of EPG, in Descartes's geometry there is no room for new problems complying with this function (unless some unconditional geometrical curves are defined in ways different to the two mentioned above). Insofar as, in Descartes's geometry as well as in EPG, problems ask for constructions, all of them comply with the constructive function.<sup>67</sup> Still, proving new constructive clauses is far less important in the former geometry than in the latter. On the other hand, in Descartes's geometry there is room for problems asking for the identification and construction of the geometrical linkages tracing the curves expressed by certain equations or sorts of equations (or for the determination of the equations expressing the curves traced by certain geometrical linkages or sorts of geometrical linkages), which are absent from EPG.

Despite these structural differences, the identity conditions of geometrical linkages and curves are, like those of EPG objects, only local and reduce to the identity conditions of the corresponding diagrams. Hence, geometrical linkages and curves also do not form a fixed domain of quantification and individual reference, in the sense explained in Section 2.1: each time one wants to refer individually to some of them, these have to be obtained, or assumed to have been obtained anew.<sup>68</sup>

<sup>66</sup> Cf. Section 2.2.

<sup>67</sup> Cf. Section 2.2.

<sup>68</sup> If one adds that, in Descartes's geometry, both geometrical linkages and geometrical curves are obtained through constructions by rulers, compass, and reiteration, and that these constructions enclose elementary constructions, one should understand why I take this geometry to be a conservative extension of EPG, in the informal sense explained in Section 2.3. But one could also wonder whether Descartes's geometry could also be said to be a conservative extension of EPG in some stronger sense. In modern logic, one says that a (formal) theory  $T^*$  is a conservative extension of a (formal) theory  $T$  if and only if the language of  $T^*$  includes that of  $T$  and any theorem of  $T^*$  formulated in the language of  $T$  is also a theorem of  $T$ . (This last requirement might not be equivalent to the requirement that any logical consequence of the axioms of  $T^*$  formulated in the language of  $T$  be also a logical consequence of the axioms of  $T$ . For example, this is not so for second-order theories. In this case, one can define two distinct senses in which a theory is a conservative extension of another one. I cannot enter these logical subtleties here.) One could then wonder whether Descartes's geometry could be said to be a conservative extension of EPG in some sense close to this one. To try to respond to this question, one can reason as follows. Take  $G$  to be a conservative extension of EPG in the informal sense explained in Section 2.3. Suppose that the solution of any problem in  $G$  is converted into a theorem asserting that the objects whose construction provides this same solution are available within  $G$  (i.e., can be obtained in a licensed



Still, the fact that geometrical curves can be defined as the curves expressed by equations allows a new modality of reference to these curves. One can refer to a plurality of curves through expressions such as ‘the curves expressed by equations of degree 3,’ or ‘the curves expressed by equations of the form  $x^{4n} = a^2(x^2 + y^2)^{2n-1}$ .’ This is quite different from referring to a plurality of geometrical objects through expressions such as ‘the triangles,’ ‘the parabolas,’ or ‘the radii of a given circle.’ The difference depends on the fact that forms of equations are equations, that is, mathematical objects in turn.<sup>69</sup> These objects are essentially different from triangles or geometrical curves. This is not only because they are not spatial, but overall because they have different identity conditions, and these conditions are such that they do form a fixed domain of quantification and individual reference, in the sense explained in Section 2.1. Moreover, forms of equations can be variously classified, to provide a large variety of classifications for the corresponding sorts of curves.

Sorts of geometrical curves become, then, mathematical objects of a new kind, essentially different from the geometrical curves themselves. An example will explain the difference. Classical geometry deals with parabolas, but there is no object in it like the totality of all the parabolas, or the parabola. There are only particular parabolas differing in the context of single arguments.<sup>70</sup> In Descartes’s algebraic geometry, there is, instead, an equation providing the canonical form of any equation expressing a parabola, and admitting a certain range of possible transformations, which is a mathematical object, as such.

It is hard to overestimate the consequences of this difference for the evolution of mathematics. There is room to say that it makes modern mathematics begin. This probably explains why historians have emphasized the connections between Descartes’s geometry

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way according to the exactness norms proper to  $G$ ). Insofar as  $G$  is a conservative extension of EPG in the previous informal sense, some of the theorems that are obtained in this way will be relative to EPG objects available within EPG (one of these theorems will assert, for example, that equilateral triangles are available within  $G$ ). It is obvious that any theorem like these corresponds to a theorem asserting that the same objects are available within EPG. But, of course, this is not enough for  $G$  to be a conservative extension of EPG in a sense close to the modern formal one. One could even argue that this does not happen any time that  $G$  is such that some EPG objects which are not available within EPG are instead available within  $G$ . But suppose now that, despite being so,  $G$  is also such that any theorem that can be proved in it about EPG objects available within EPG (and then also within  $G$  itself) might also be proved in EPG. In this case, one could say that  $G$  is, after all, a conservative extension of EPG in another sense close to the modern logical one. An interesting question concerning Descartes’s geometry is whether it is a conservative extension of EPG in this last sense.

<sup>69</sup> This depends on what Manders has called ‘representational unresponsiveness’ of algebraic literal notation [Manders 2008, 73]: the fact that a literal symbol entering into this notation and denoting a particular object expresses no specific feature of this same object. This is because the same symbol used to denote a particular geometric object, or a perfectly analogous symbol, can also denote other objects of the same sort, or stand for any objects of the same sort. It is not the same for the literal symbols used in classical geometry to denote geometrical objects, that is, for symbols such as ‘AB’ used to denote a segment. The reason is obvious: these symbols are used to denote the objects represented by certain diagrams (either actually drawn or imagined: cf. footnote 18) to which they are relative, and diagrams do not enjoy representational unresponsiveness.

<sup>70</sup> One could say that, though there is nothing such as the parabola, there is at least the kind of parabolas, understood as a particular kind of conics. Taken as a single object, this is not, however, a mathematical object, since there is no way to mathematically operate on it, as such. At most, this is the linguistic hypostasis of a mathematical concept.



and modern mathematics much more than they have insisted on its genetic relations with EPG and classical geometry. I have tried, instead, to show some of these relations, by emphasizing structural analogies and differences, and using them to account for Descartes's attitude towards geometrical exactness. Though I have presented this attitude as a foundational one, I did not mean to undermine the centrality of Descartes's concern for problem-solving. What I have suggested is rather a way to articulate this concern with his foundational program.

### Acknowledgements

I thank Andrew Arana, Rebekah Arana, Carlos Alvarez, June Barrow-Green, Henk Bos, Bernard Cache, Mario Carpo, Olivia Chevallier, Annalisa Coliva, Davide Crippa, Mary Domski, Jacques Dubucs, Massimo Galuzzi, Pierluigi Graziani, Niccolò Guicciardini, Robin Hartshorne, Ignacio Jané, Vincent Jullien, Antoni Malet, Paolo Mancosu, Sébastien Maronne, Jean-Michel Salanskis, David Rabouin, André Warusfel, and two anonymous referees, for valuable comments and/or suggestions.

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